7.1. Limit of a function.

Let f be a real function defined on a domain $D \subset \mathbb{R}$. In order that f may have a limit $l(\in \mathbb{R})$ at a point c, for x sufficiently close to c, $f(x)$ should be arbitrarily close to l. For this to be meaningful, it is necessary that c be a limit point of the domain D. Keeping this requirement in view, we give the formal definition.

Definition. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let c be a limit point of D. A real number l is said to be a *limit of f at c* if corresponding to any neighbourhood V of l there exists a neighbourhood W of c such that $f(x) \in V$ for all $x \in [W - \{c\}] \cap D$.
This is expressed by the symbol $\lim_{x \to c} f(x) = l$.

Equivalent definition. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let c be a limit point of D. A real number l is said to be a *limit of f at c* if corresponding to a pre-assigned positive ϵ there exists a positive δ such that

$l - \epsilon < f(x) < l + \epsilon$ for all $x \in N'(c, \delta) \cap D$,
where $N'(c, \delta) = \{x \in \mathbb{R} : 0 < |x - c| < \delta\} = (c - \delta, c + \delta) - \{c\}$.

Note 1. In order that we may enquire if $\lim_{x \to c} f(x)$ exists, c must be a *limit point* of the domain D of the function f.

Note 2. The definition demands that all values of f in some deleted δ-neighbourhood $N'(c, \delta)$ contained in D must lie in the chosen ϵ-neighbourhood of l. It does not matter whether c belongs to D or not. Even if $c \in D$, $f(c)$ need not lie in the ϵ-neighbourhood of l.

Theorem 7.1.1. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $c \in D'$. Then f can have at most one limit at c.

Proof. Suppose, on the contrary, there exist two different limits, l, m of the function f at c.

Since $l \neq m$, we assume $m > l$, without loss of generality. Let $\epsilon = \frac{m-l}{2} > 0$. Then the neighbourhoods $(l - \epsilon, l + \epsilon)$ and $(m - \epsilon, m + \epsilon)$ are disjoint.

Since l is a limit of f at c, there exists a positive δ_1 such that
$l - \epsilon < f(x) < l + \epsilon$ for all $x \in N'(c, \delta_1) \cap D$.
Since m is a limit of f at c, there exists a positive δ_2 such that
$m - \epsilon < f(x) < m + \epsilon$ for all $x \in N'(c, \delta_2) \cap D$.

Let $\delta = \min\{\delta_1, \delta_2\}$.

Then $l - \epsilon < f(x) < l + \epsilon$ and $m - \epsilon < f(x) < m + \epsilon$ for all $x \in N'(c, \delta) \cap D$. This is a contradiction, since the neighbourhoods $(l-\epsilon, l+\epsilon)$ and $(m - \epsilon, m + \epsilon)$ are disjoint.

Therefore $l = m$ and the theorem is done.

Worked Examples.

1. Show that $\lim_{x \to 2} f(x) = 4$, where $f(x) = \frac{x^2-4}{x-2}, x \neq 2$.

Here the domain D of f is $\mathbb{R} - \{2\}$. 2 is a limit point of D.
When $x \in D, | f(x) - 4 | = | \frac{x^2-4}{x-2} - 4 | = | x - 2 |$.
Let us choose $\epsilon > 0$.
$| f(x) - 4 | < \epsilon$ whenever $| x - 2 | < \epsilon$ and $x \in D$, i.e., for all $x \in D$ satisfying $0 < | x - 2 | < \epsilon$.
Therefore $| f(x) - 4 | < \epsilon$ for all $x \in N'(2, \delta) \cap D$ [taking $\delta = \epsilon$].
So we have $\lim_{x \to 2} f(x) = 4$.

2. Show that $\lim_{x \to 2} f(x) = 4$, where $f(x) = \frac{x^2-4}{x-2}, x \neq 2$
$= 10, \ x = 2$.

Here the domain D of f is $\mathbb{R}$. 2 is a limit point of D.
When $x \in D - \{2\}, | f(x) - 4 | = | \frac{x^2-4}{x-2} - 4 | = | x - 2 |$.
Let us choose $\epsilon > 0$.
$| f(x) - 4 | < \epsilon$ whenever $| x - 2 | < \epsilon$ and $x \neq 2$.
Therefore $| f(x) - 4 | < \epsilon$ for all $x \in N'(2, \delta) \cap D$ [taking $\delta = \epsilon$].
So we have $\lim_{x \to 2} f(x) = 4$.

3. Show that $\lim_{x \to 0} f(x) = 0$ where $f(x) = \sqrt{x}, x \geq 0$.

Here the domain D of f is $\{x \in \mathbb{R} : x \geq 0\}$. 0 is a limit point of D.
Let us choose $\epsilon > 0$.
When $x \geq 0, | f(x) - 0 | = \sqrt{x}$.
Therefore $| f(x) - 0 | < \epsilon$ for all x satisfying $0 < x < \epsilon^2$, i.e., for all $x \in N'(0, \delta) \cap D$ [taking $\delta = \epsilon^2$].
So we have $\lim_{x \to 0} f(x) = 0$.

Note. Here $N'(0, \delta) \cap D = (0, \delta)$, since $D = \{x \in \mathbb{R} : x \geq 0\}$.