'समानो मन्त्रः समितिः समानी'

UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 6th Semester Examination, 2024

CC13-MATHEMATICS

RING THEORY AND LINEAR ALGEBRA-II

(REVISED SYLLABUS 2023 / OLD SYLLABUS 2018)

Time Allotted: 2 Hours Full Marks: 60

The figures in the margin indicate full marks.

GROUP-A

1. Answer any ***four*** questions: $3\times4=12$

(a) Prove that in an Integral domain R, every prime element is irreducible.

(b) Find the minimal polynomial of the matrix $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$.

(c) Find the dual basis of the basis $\beta=\{(1,0,1),\ (1,2,1),\ (0,0,1)\}$ of $\mathbb{R}^3$.

(d) Test for the diagonalizability of the matrix $\begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix}$ in $M_2(\mathbb{R})$.

(e) Show that $1-i$ is irreducible in $\mathbb{Z}[i]$.

(f) If in an inner product space $\|\alpha+\beta\|=\|\alpha\|+\|\beta\|$ holds, prove that the vectors α and β are linearly dependent.

GROUP-B

Answer any *four* questions $6\times4=24$

2. (a) Show that the integral domain $\mathbb{Z}$ is Euclidean domain. 4+2

(b) Show that $\sqrt{-3}$ is a prime element in the integral domain $\mathbb{Z}[\sqrt{-3}]$.

3. (a) Use gram Schmidt process to obtain an orthogonal basis from the basis set $\{(1,1,1),\ (1,1,1),\ (1,3,4)\}$. 4+2

(b) Let β be a basis for a finite dimensional inner product space. Prove that if $\langle x,\ z\rangle=0$ for all $z\in\beta$, then $x=0$.

4. Let R be a UFD and $f(x)\in R[x]$. Show that $f(x)$ is irreducible over R if and only if $f(x+a)$ is irreducible over R for any $a\in R$. 6

5. Let W be a subspace of $\mathbb{R}^4$ spanned by $\alpha_1 = (1, 2, -3, 4)$, $\alpha_2 = (0, 1, 4, -1)$. Find the annihilator W^0 of W, and a basis of W^0. 6

6. Prove that if k is a field, then $k[x]$ is a Euclidean domain. 6

7. Each eigen value of a real orthogonal matrix has unit modulus. — Explain. 6

GROUP-C

Answer any *two* questions 12×2 = 24

8. (a) Find the minimal polynomial of the matrix 6

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

(b) Let V be a finite dimensional inner product space. If T and S are linear operators on V, then show that $(S+T)^* = S^* + T^*$ and $(T^*)^* = T$, where T^* is the adjoint of T. 6

9. (a) Show that $I = \{(a, 0) : a \in \mathbb{Z}\}$ is a prime ideal but not a maximal ideal of the ring $\mathbb{Z} \times \mathbb{Z}$. 4

(b) If P be a non-zero non-unit element in a PID R, then prove that the following statements are equivalent: 8

(i) P is a prime element in R.

(ii) P is an irreducible element in R.

(iii) $\langle P \rangle$ is a nonzero maximal ideal of R.

(iv) $\langle P \rangle$ is a nonzero prime ideal of R.

10.(a) Let T be a linear operator on $\mathbb{R}^3$ defined by $T(a, b, c) = (a+b,\ b+c,\ 0)$. Show that xy-plane and the x-axis are T-invariant subspace of $\mathbb{R}^3$. 6

(b) Let $T: V \to V$ be a linear mapping and $\lambda \in F$ be an eigen value of T. Then prove that $V_\lambda = \{v \in V :\ T_v = \lambda V\}$ is a subspace of V. 6

11.(a) Let V and W be vector spaces, and let S be a subset of V. 5+5

Define $S^0 = \{T \in L(V, W) : T(\alpha) = \theta,\ \forall \alpha \in S\}$.

Prove the following:

(i) If S_1 and S_2 are subset of V and $S_1 \subseteq S_2$, then $S_2^0 \subseteq S_1^0$.

(ii) If V_1 and V_2 are subspaces of V, then $(V_1 + V_2)^0 = V_1^0 \cap V_2^0$.

(b) Prove that $2+11i$ and $2-7i$ are relatively prime in the integral domain $\mathbb{Z}[i]$. 2

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