'समानो मन्त्रः समितिः समानी'

UNIVERSITY OF NORTH BENGAL

B.Sc. Major 2nd Semester Examination, 2024

UMATMAJ12002-MATHEMATICS

CALCULUS AND GEOMETRY

Time Allotted: 2 Hours 30 Minutes | Full Marks: 60

The figures in the margin indicate full marks.

GROUP-A

1. Answer any ***four*** questions: $3\times 4 = 12$

(a) If $\lim\limits_{x\to 0}\dfrac{\sin 2x + a\sin x}{x^3}$ be finite, find the value of a and the limit. 3

(b) Determine the values of h and g so that the equation 3

$$x^2 - 2hxy + 4y^2 + 2gx - 12y + 9 = 0$$

may represent a conic having no centre.

(c) What does the equation $11x^2 + 16xy - y^2 = 0$ become when the axes are rotated through an angle $\tan^{-1}(1/2)$? 3

(d) Find the asymptotes of the curve $y = \log(x-1)$. 3

(e) If $I_n = \int\limits_0^{\pi/2} \sin^n x\,dx$, where n is a positive integer, then prove that $I_n = \dfrac{n-1}{n} I_{n-2}$ for $n > 2$. 3

(f) Find the equation of the sphere through the circle $x^2 + y^2 + z^2 = 25$, $x + 2y - z + 2 = 0$ and the point (1, 1, 1). Also find its centre and radius. 3

GROUP-B

2. Answer any ***four*** questions: $6\times 4 = 24$

(a) If $y = \sin(m\sin^{-1}x)$, show that $(1-x^2)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0$. Hence find $y_n(0)$. 6

(b) Reduce the equation 6

$$x^2 + y^2 + z^2 - 2xy - 2yz + 2zx + x - 4y + z + 1 = 0$$

to its canonical form and determine the type of the quadric represented by it.

(c) If m is a positive integer, then prove that 6

$$\int_0^{\pi/2} \cos^m x \sin x\, dx = \frac{1}{2^{m+1}}\left[2+\frac{2^2}{2}+\frac{2^3}{3}+\cdots\cdots+\frac{2^m}{m}\right]$$

(d) Show that the straight line $r\cos(\theta-\alpha)=p$ touches the conic $\frac{l}{r}=1+e\cos\theta$, if $(l\cos\alpha-ep)^2+l^2\sin^2\alpha=p^2$. 6

(e) Find the volume of the solid formed by revolving one loop of the curve $r^2=a^2\cos 2\theta$ about the line $\theta=\pi/2$. 6

(f) A plane passes through a fixed point (p, q, r) and cut the axes at A, B, C. Show that the locus of the centre of sphere $OABC$ is $\frac{p}{x}+\frac{q}{y}+\frac{r}{z}=2$, where O is the origin. 6

GROUP-C

Answer any *two* questions $12\times 2=24$

3. (a) Find the pedal equation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with respect to the centre as pole. 6

(b) Find the envelope of the family of circles whose centre lie on the rectangular hyperbola $xy=c^2$ and pass through the centre of the hyperbola. 6

4. (a) Show that the plane $6x+4y+3z-12=0$ intersects the hyperboloid $\frac{x^2}{4}+\frac{y^2}{9}-\frac{z^2}{16}=1$ in two generators. 6

(b) If PSP' and QSQ' be any two perpendicular focal chords of a conic, then prove that $\frac{1}{SP\cdot SP'}+\frac{1}{SQ\cdot SQ'}=$ a constant. 6

5. (a) Let ρ_1 and ρ_2 be the radii of curvature at the ends P and Q of conjugate diameters CP and CQ respectively, of the ellipse $(x^2/a^2)+(x^2/b^2)=1$. Show that $\rho_1^{2/3}+\rho_2^{2/3}=(a^2+b^2)/(ab)^{2/3}$, where C is the centre of the ellipse. 6

(b) Prove that the area common to the circles $r=a\sqrt{2}$ and $r=2\cos\theta$ is $a^2(\pi-1)$. 6

6. (a) Find the equation of the cylinder which passes through the point $(3,-1,1)$ and has the axis $\frac{x-1}{2}=\frac{y+3}{-1}=\frac{z-2}{1}$. 6

(b) Find the equation of the generating lines of the hyperboloid $\frac{x^2}{4}+\frac{y^2}{9}-\frac{z^2}{16}=1$ passing through the point $(2,-1, 4/3)$. 6

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