UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 5th Semester Examination, 2024

DSE-P2-MATHEMATICS

(REVISED SYLLABUS 2023 / OLD SYLLABUS 2018)

Time Allotted: 2 Hours | Full Marks: 60

The figures in the margin indicate full marks.

The question paper contains DSE2A and DSE2B. Candidates are required to answer any *one* from the *two* DSE2 courses and they should mention it clearly on the Answer Book.

DSE2A

NUMBER THEORY

GROUP-A

1. Answer any *four* questions: $3\times 4=12$

(a) Find an inverse of 17 under modulo 23. 3

(b) Prove that for any $n\in\mathbb{N}$, 3

$$2^{4n}\equiv 1\ (\text{mod}\ 15)$$

(c) Determine all the solutions in integers of $24x+138y=18$. 3

(d) Prove that, for two distinct odd primes p and q, 3

$$(p/q)=\begin{cases}(q/p) & \text{if}\ \ p\equiv 1\ (\text{mod}\ 4)\ \text{or}\ \ q\equiv 1\ (\text{mod}\ 4)\\ -(q/p) & \text{if}\ \ p\equiv q\equiv 3\ (\text{mod}\ 4)\end{cases}$$

(e) Find $\gcd(1769, 2378)$. 3

(f) If p_n denotes the n^{th} prime, prove by mathematical induction that $p_n\le 2^{2^{n-1}}$. 3

GROUP-B

2. Answer any *four* questions: $6\times 4=24$

(a) (i) Define Pythagorean triple. 2+4

(ii) If x, y, z is a Pythagorean triple such that $\gcd(x, y, z)=1$, then show that one of the integers x or y is even, while the other is odd.

(b) Find the remainder when the following sum is divided by 4: 6

$$1^5+2^5+3^5+\cdots\cdots+100^5$$

(c) Prove that a prime number p is a Gaussian prime iff it is not the sum of two squares. 6

(d) Let m and n be positive integers such that $m > 2$, $n > 2$ and $\gcd(m, n) = 1$. Prove that mn has no primitive roots. 6

(e) Solve by Chinese Remainder theorem: 6

$$x \equiv 1 \pmod 3$$
$$x \equiv 2 \pmod 5$$
$$x \equiv 3 \pmod 7$$

(f) If p be a prime, prove that for any integer a, $a^p \equiv a \pmod p$. 6

GROUP-C

3. Answer any *two* questions: $12 \times 2 = 24$

(a) Let p be an odd prime and a be an integer with $\gcd(a, p) = 1$. Consider the set 7+5

$$S = \left\{a,\ 2a,\ 3a,\ \ldots\ldots,\ \left(\frac{p-1}{2}\right)a\right\}$$

If n denotes the number of integers in S having remainder greater than $p/2$, upon division by p, then prove that $(a/p) = (-1)^n$.

Further, if a is an odd integer, then prove that

$$(a/p) = (-1)^{\sum_{k=1}^{\frac{p-1}{2}}\left[\frac{ka}{p}\right]}$$

(b) State Fermat's Two square theorem. Show that it is necessary and sufficient. 2+5+5

(c) (i) Prove that there are infinite number of primes of the form $4k+3$. 4+4+4

(ii) If $ab \equiv cd \pmod n$ and $b \equiv d \pmod n$ with $\gcd(b, n) = 1$, then prove that $a \equiv c \pmod n$.

(iii) Show that $5^{38} \equiv 4 \pmod{11}$.

(d) (i) Let p be an odd prime and a be an integer with $\gcd(a, p) = 1$. Then prove that a is a quadratic residue of p iff $a^{\frac{(p-1)}{2}} \equiv 1 \pmod p$. 6

(ii) Define gcd in the set of Gaussian integers. Find the gcd of $-3+11i$ and $8-i$ in $\mathbb{Z}[i]$. Also find $x, y \in \mathbb{Z}[i]$ such that 1+5

$$\gcd(-3+11i,\ 8-i) = (-3+11i)x + (8-i)y$$

DSE2B
MECHANICS
GROUP-A

1. Answer any *four* from the following: $3 \times 4 = 12$

(a) A uniform cubical box of edge a is placed on the top of a fixed sphere. Show that the least radius of the sphere for which the equilibrium will be stable is $\frac{a}{2}$. 3

(b) Three forces P, Q, R act along the sides of the triangle formed by the lines $x+y=1$, $y-x=1$, $y=2$. Obtain the line of action of their resultant. 3

(c) A particle moves freely in a periodic path given by $y^2 = 4ax$ under a force which is always perpendicular to its axis. Find the law of force. 3

(d) Write down the Kepler's laws of planetary motion. 3

(e) The lengths AB and AD of the sides of a rectangle $ABCD$ are $2a$ and $2b$; show that the inclination to AB of one of the principal axes at A is $\frac{1}{2}\tan^{-1}\frac{3ab}{2(a^2-b^2)}$. 3

(f) Define equimomental system. Under what conditions two systems will be equimomental? 3

GROUP-B

2. Answer any *four* from the following: $6\times 4 = 24$

(a) State and prove the principle of virtual work for any system of non-coplanar forces acting on a rigid body. 6

(b) A solid cone of height h and semi-vertical angle α, is placed with its base against a smooth wall which is vertical and is suspended by a string attached to its vertex and to a point in the wall. Show that the greatest possible length of the string is $h\sqrt{1+\frac{16}{9}\tan^2\alpha}$. 6

(c) A straight smooth tube revolves with constant angular velocity ω in a horizontal plane about one extremity which is fixed. If initially a particle inside it be at a distance a from a fixed end and moving with constant velocity V along the tube, then show that its distance at time t is $a\cosh\omega t + \frac{V}{\omega}\sinh\omega t$. 6

(d) A particle describes an ellipse of eccentricity e about a centre of force at a focus. When the particle is at one end of a major axis, its velocity is doubled. Prove that the new path is a hyperbola of eccentricity $\sqrt{9-8e^2}$. 6

(e) Define compound pendulum and obtain the length of corresponding simple equivalent pendulum. State the conditions for the minimum time for oscillation of the compound pendulum. 6

(f) A uniform rod AB is freely movable on a rough inclined plane whose inclination to the horizon is 'i' and whose co-efficient of friction is μ about a smooth pin fixed through the end A. The rod is held in the horizontal position in the plane and allowed to fall from this position. If θ be the angle through which it falls from rest, then show that $\frac{\sin\theta}{\theta} = \mu\cot i$. 6

GROUP-C

3. Answer any *two* from the following: $12\times 2 = 24$

(a) (i) Forces P, Q, R act along any three mutually perpendicular generators of the same system of the surface $x^2+y^2 = 2(z^2+a^2)$, the positive direction of the forces being towards the same side of x-y plane. Prove that the pitch of the equivalent wrench is $2a\cdot\frac{(PQ+QR+RP)}{P^2+Q^2+R^2}$. 6

(ii) Find the C.G. of a hemisphere whose density varies as the distance from a point on its plane edge. 6

(b) (i) A particle is subjected to a central force per unit mass equal to $\mu\{2(a^2+b^2)u^2-3a^2b^2u^7\}$ and is projected at a distance a with a velocity $\frac{\sqrt{\mu}}{a}$ in a direction of right angle to the distance. Find the path described by the particle. 6

(ii) Obtain the equation of motion of a particle moving in a central orbit under a central force F and then deduce the differential equation of the orbit in the form $\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2}$, where the symbols have their usual meaning. 6

(c) (i) Obtain the equations of motion of rigid body in two dimensions under finite forces. 6

(ii) A uniform rod is held at an inclination α to the horizontal with one end in contact with the horizontal table whose coefficient of friction is μ. If it be then released, then show that, it will commence to stable, if $\mu<\frac{3\sin\alpha\cos\alpha}{1+3\sin^2\alpha}$. 6

(d) (i) If a planet suddenly stopped in its orbit, supposed circular, show that it would fall into the Sun in a time which is $\frac{\sqrt{2}}{8}$ times the period of the planet's revolution. 6

(ii) Prove that a uniform triangular lamina of mass M is equimomental with three equal masses $\frac{M}{3}$ placed at the mid points of its sides. 6

——×——