'समानो मन्त्रः समितिः समानी'

UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 2nd Semester Examination, 2024

CC3-MATHEMATICS

REAL ANALYSIS

(REVISED SYLLABUS 2023 AND OLD SYLLABUS 2018)

Time Allotted: 2 Hours | Full Marks: 60

The figures in the margin indicate full marks.

GROUP-A

1. Answer any ***four*** questions: $3\times4 = 12$

(a) Prove that the intersection of a finite number of open sets in $\mathbb{R}$ is an open set. 3

(b) Show that $\left\{\frac{3n+1}{n+2}\right\}$ is a bounded sequence. 3

(c) Two real numbers a and b are such that for all $\varepsilon > 0$, $|a-b| < \varepsilon$ holds. Show that $a = b$. 3

(d) Show that $\mathbb{Z}$ is an enumerable set. 3

(e) Examine the convergence of the series $x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\cdots$, $x > 0$. 3

(f) Give an example of an open cover of the set (0, 1) which does not have a finite sub-cover with explanation. 3

GROUP-B

Answer any *four* questions $6\times4 = 24$

2. State and prove Bolzano-Weierstrass' theorem for subsets of $\mathbb{R}$. 6

3. Test the convergence of the series $\frac{1}{4} + \left(\frac{1}{4}\right)^{1+\frac{1}{3}} + \left(\frac{1}{4}\right)^{1+\frac{1}{3}+\frac{1}{5}} + \cdots\cdots$. 6

4. Show that finite union of compact subset of $\mathbb{R}$ is compact. 6

5. State Sandwich Theorem. Show that the sequence $\{u_n\}$ converges to 0 where 6

$$u_n = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots\cdots + \frac{1}{(n+n)^2}$$

6. If $\{S_n\}$ is a sequence such that $S_{n+1}=\sqrt{\dfrac{ab^2+S_n^2}{a+1}}$, $b>a$, $\forall\, n\geq 1$ and $S_1=a>0$. Then show that the sequence $\{S_n\}$ is an increasing bounded above sequence and $\lim\limits_{n\to\infty} S_n = b$. 6

7. Define open set. Prove that union of finite number of open sets is an open set. Is the union of an arbitrary collection of open sets is open? 1+4+1

GROUP-C

Answer any *two* questions 12×2 = 24

8. (a) Show that finite product of enumerable sets is enumerable. Also show that enumerable product of finite sets may not be enumerable. 4+4

(b) A sequence $\{x_n\}$ is such that every subsequence of $\{x_n\}$ has a subsequence that converges to 0. Show that $\lim x_n = 0$. 4

9. (a) Show that the set of all real numbers in the closed interval [0, 1] is not countable. 6

(b) Show that a derived set is always closed. 4

(c) Find $\overline{\lim}\, u_n$ and $\underline{\lim}\, u_n$, where $u_n = \dfrac{n}{4} - \left[\dfrac{n}{4}\right]$. 1+1

10. (a) Let $S \subset \mathbb{R}$ be non-empty. For $a \in \mathbb{R}$, define $d(a, S) = \inf\{|a-x| : x \in S\}$. Show that $a \in \overline{S}$ iff $d(a, S) = 0$. 4

(b) What is completeness property of $\mathbb{R}$? 2

(c) Let $S = \left\{\dfrac{1}{2^m} + \dfrac{1}{2^n} : m,\ n \in \mathbb{N}\right\}$. Find the derived set of S. 6

11. (a) If $\{x_n\}$ is a sequence of positive real numbers converges to l, then show that 3

$$\lim_{n\to\infty} \sqrt[n]{x_1 x_2 \cdots\cdots x_n} = l$$

(b) Test the convergence of the series $1 + \dfrac{2x}{2!} + \dfrac{3^2x^2}{3!} + \dfrac{4^3x^3}{4!} + \cdots\cdots$. 4

(c) State and prove Leibnitz's test for convergence of an alternating series. 5

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