'समानो मन्त्रः समितिः समानी'

UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 1st Semester Examination, 2024

CC2-MATHEMATICS

ALGEBRA

(REVISED SYLLABUS 2023 / OLD SYLLABUS 2018)

Time Allotted: 2 Hours — Full Marks: 60

The figures in the margin indicate full marks.

GROUP-A

1. Answer any ***four*** questions: $3\times4=12$

(a) If $x^3+3px+q$ has a factor of the form $(x-\alpha)^2$, then show that $q^2+4p^3=0$. 3

(b) If $a_1<a_2<\cdots\cdots<a_n$ be n positive real numbers, show that 3

$$a_1<\frac{a_1^2+\cdots\cdots+a_n^2}{a_1+\cdots\cdots+a_n}<a_n$$

(c) Check the consistency of the system of linear equations: 3

$$x-2y+2z=1$$
$$2x+y-z=2$$
$$5x+5y-5z=6$$

(d) Apply Descartes' rule of signs to ascertain the minimum number of complex roots of the equation $x^7-3x^3+x^2=0$. 3

(e) Find the product of all the values of $(1+i)^{4/5}$. 3

(f) Find the rank of the matrix $A=\begin{pmatrix}1 & -2 & -1\\ -1 & 2 & 1\end{pmatrix}$. 3

GROUP-B

Answer any *four* questions $6\times4=24$

2. Solve the biquadratic equation by Ferrari's method: $x^4+2x^3-7x^2-8x+12=0$ 6

3. If $x=\log\tan(\pi/4+\theta/2)$, where θ is real, prove that, $\theta=-i\log\tan(\pi/4+i\,x/2)$. 6

4. If $\lambda\neq-14$, then prove that the system of equations: 6

$$5x+2y-z=1$$
$$2x+3y+4z=7$$
$$4x-5y+\lambda z=\lambda-5$$

has a unique solution at (0, 1, 1).

5. If $a_1, a_2, \cdots\cdots, a_n$ and $t_1, t_2, \cdots\cdots, t_n$ be two list of real numbers, then show that $(a_1t_1 + a_2t_2 + \cdots\cdots + a_nt_n)^2 \le (a_1^2 + a_2^2 + \cdots\cdots + a_n^2)(t_1^2 + t_2^2 + \cdots\cdots + t_n^2)$ and the equality holds when $\frac{a_1}{t_1} = \frac{a_2}{t_2} = \cdots\cdots = \frac{a_n}{t_n}$. 6

6. Find the equation whose roots are the squared differences of the roots of the cubic equation $x^3 - 13x - 12 = 0$. 6

7. Find a non-singular matrix P such that $P^T AP$ is the normal form of A under congruence, where $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$. Also find the rank, index and signature of P. 6

GROUP-C

Answer any *two* questions $12\times2 = 24$

8. (a) Expand, with the help of De Moivre's theorem, $\cos 7\theta$ in terms of $\cos\theta$. 6+6

(b) If $\alpha = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ and if p is prime to n, prove that

$$1 + \alpha^p + \alpha^{2p} + \cdots\cdots + \alpha^{(n-1)p} = 0$$

9. (a) If α, β, γ be the roots of the equation $x^3 + px^2 + qx + r = 0$, find the value of 4+4+4

$$\left(\frac{1}{\beta} + \frac{1}{\gamma} - \frac{1}{\alpha}\right)\left(\frac{1}{\gamma} + \frac{1}{\alpha} - \frac{1}{\beta}\right)\left(\frac{1}{\alpha} + \frac{1}{\beta} - \frac{1}{\gamma}\right).$$

(b) If 1, $\alpha_1, \alpha_2, \cdots\cdots, \alpha_{n-1}$ be n distinct roots of the equation $x^n - 1 = 0$, then prove that

$$(1-\alpha_1)(1-\alpha_2)\cdots\cdots(1-\alpha_{n-1}) = n$$

(c) Prove that the roots of the equation $\frac{1}{x-1} + \frac{2}{x-2} + \frac{3}{x-3} = x$ are all real.

10.(a) Determine the conditions on a and b, for which the system of equations 6+6

$$x + 2y + z = 1$$
$$2x + y + 3z = b$$
$$x + ay + 3z = b + 1$$

has (i) a unique solution, (ii) no solution, (iii) many solutions.

(b) Use Cayley Hamilton Theorem to find the inverse of $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix}$.

11.(a) For distinct real numbers a, b, c with $a + b + c = 1$, prove that 4+4+4

$$8abc < (1-a)(1-b)(1-c) < \frac{8}{27}$$

(b) Consider the subset $A = \{x \in \mathbb{R}: 0 < x < 1\}$ of $\mathbb{R}$. If a mapping $f : A \to \mathbb{R}$ is defined by $f(x) = \frac{2x-1}{1-|2x-1|}$, $x \in A$, then show that f is bijective.

(c) If $n\ (>1)$ be a positive integer, prove that $(n+1)^{n-1}(n+2)^n > 3^n(n!)^2$.

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