'समानो मन्त्रः समितिः समानी'

UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 5th Semester Examination, 2024

CC11-MATHEMATICS

GROUP THEORY-II

(REVISED SYLLABUS 2023 / OLD SYLLABUS 2018)

Time Allotted: 2 Hours | Full Marks: 60

The figures in the margin indicate full marks.

GROUP-A

1. Answer any ***four*** questions: $3\times4=12$

(a) Find all Sylow 2-subgroups of S_3. 3

(b) For a group G if $f:G\to G$, defined by $f(x)=x^2, \forall x\in G$, is an automorphism, does it imply that G is commutative. 3

(c) Let G be a group and S be a G-set. Show that $\forall a\in S$, the subset $G_a=\{g\in G\,;\, ga=a\}$ is a subgroup of G. 3

(d) Find the number of elements of order 7 in $\mathbb{Z}_4\times\mathbb{Z}_7$. 3

(e) Let G be a finite group that has only two conjugate classes. Show that order of the group G is 2. 3

(f) Show that $\operatorname{Inn}(G)$ is subgroup of $\operatorname{Aut}(G)$, where G is a group. 3

GROUP-B

Answer any *four* questions $6\times4=24$

2. Let G be a simple group of order 168. Find the number of subgroups of order 7. 6

3. Let G be a finite group and S be a G-set. Prove that $|S|=\sum_{a\in A}[G:G_a]$, where A is a subset of S containing exactly one element from each orbit $[a]$. 6

4. State and prove Sylow's Third Theorem. 6

5. (a) Prove that if G is a finite group, then G is a p-group iff $O(G)=p^n$ for some non-negative integer n. 4+2

(b) Write down the class equation of S_3.

6. Show that for any prime p, there exists only two non-isomorphic groups of order p^2. 6

7. Find Aut (K_4) and Aut ($\mathbb{Z}_4$). 6

GROUP-C

Answer any *two* questions $12\times2=24$

8. (a) Show that any group of order pq where p, q are primes, $p>q$ and q does not divide $p-1$, is cyclic. 6+6

   (b) Show that a group of prime order must always have a non-trivial centre.

9. (a) Let $S=\{x_1 x_2 x_3 \cdots\cdots x_n;\ \ n\geq 1$, each x_i is a commutator in $G\}$ be the collection of all finite products of commutators of a group G. Show that S is a normal subgroup of G. 6+6

   (b) Show that $|\text{Aut}(\mathbb{Z}_2\times\mathbb{Z}_2)|=6$.

10. (a) Prove that no group of order 30 is simple. 4+4+4

    (b) Let H be a normal subgroup of a group G. Define $\sigma: G\times H\to H$ by $\sigma(g,h)=ghg^{-1}\ \ \forall\,(g,h)\in G\times H$. Prove that σ defines an action of G on H.

    (c) Prove that $\mathbb{Z}_m\times\mathbb{Z}_n$ is cyclic iff $\gcd(m,n)=1$. Is $\mathbb{Z}\times\mathbb{Z}$ cyclic? Justify your answer.

11. (a) Find all the abelian groups of order 360. 4+6+2

    (b) Show that there exists only two groups of order 4 upto isomorphism.

    (c) State Index Theorem.

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