'समानो मन्त्रः समितिः समानी'

UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 5th Semester Examination, 2023

DSE-P1-MATHEMATICS

(REVISED SYLLABUS 2023)

Time Allotted: 2 Hours | Full Marks: 60

The figures in the margin indicate full marks.

The question paper contains DSE1A and DSE1B. Candidates are required to answer any *one* from the *two* DSE1 courses and they should mention it clearly on the Answer Book.

DSE1A

PROBABILITY AND STATISTICS

GROUP-A

1. Answer any *four* questions: $3\times4=12$

(a) If two events A and B are such that $P(A+B)=\frac{3}{4}$, $P(AB)=\frac{1}{4}$, $P(\overline{A})=\frac{2}{3}$, then find $P(\overline{A}B)$.

(b) Show that Tchebycheff's inequality that in 2000 throws with a coin, the probability that the number of heads lies between 900 and 1100 is atleast $\frac{19}{20}$.

(c) State weak law of large numbers.

(d) Let T_1 and T_2 be two unbiased estimators of the parameter θ. Under what condition aT_1+bT_2 will be an unbiased estimator of θ?

(e) Find the characteristic function of a Binomial distribution with parameters n and p.

(f) Let X be a random variable following Poisson distribution. If $P(X=1)=P(X=2)$, find $E(X)$.

GROUP-B

2. Answer any *four* questions: $6\times4=24$

(a) From a pack of 52 cards, an even number of cards is drawn. Show that the probability that these consist of half of red and half of black is

$$\frac{\left[\frac{52!}{(26!)^2}-1\right]}{(2^{51}-1)}$$

(b) Find the maximum likelihood estimate of the parameter λ of a continuous population having the density function $f(x)=\lambda x^{\lambda-1},\ 0<x<1\ ,\ \lambda>0$.

(c) (i) Prove that the second order moment of a random variable X is minimum when taken about its mean. 3+3

(ii) If $X_1, X_2, \cdots\cdots, X_n$ be a set of mutually independent random variables having characteristic functions $\chi_1(t), \chi_2(t), \cdots\cdots, \chi_n(t)$ respectively, prove that the characteristic function $\chi(t)$ of their sum S_n is given by $\chi(t)=\chi_1(t)\cdot\chi_2(t)\cdots\cdots\chi_n(t)$.

(d) If m and μ_r denote the mean and central r-th moment of a Poisson distribution, then prove that $\mu_{r+1} = rm\mu_{r-1} + \dfrac{md\mu_r}{dm}$.

(e) If $a(\neq 0)$, $c(\neq 0)$, b, d are constants, prove that $\rho(aX+b,\ cY+d) = \dfrac{ac\,\rho(X,Y)}{|a||c|}$.

(f) If $X_1, X_2, \cdots\cdots, X_n$ are mutually independent random variables and each X_i has uniform distribution over the interval (a,b), then find the density function of the random variable U, given by $U = \min\{X_1, X_2, \cdots\cdots, X_n\}$.

GROUP-C

3. Answer any *two* questions: $12\times 2 = 24$

(a) (i) A drug is given to 10 patients, and the increments in their blood pressure were recorded to be 3, 6, –2, 4, –3, 4, 6, 0, 0, 2. Is it reasonable to believe that the drug has no effect on the change of blood pressure? Test at 5% significance level, assuming the population to be normal. 7

(ii) The joint density function of the random variable X, Y is given by 5

$$f(x, y) = 2\ (0 < x < 1,\ 0 < y < x)$$

Find the marginal and conditional density functions.

Compute $P\left(\frac{1}{4} < X < \frac{3}{4} \middle| Y = \frac{1}{2}\right)$.

(b) (i) The joint probability density function of two random variable X and Y is 6

$$f(x, y) = 8xy\ ,\ 0 \le x \le y\ ,\ 0 \le y \le 1$$
$$= 0\ ,\quad \text{otherwise}$$

Examine whether X and Y are independent. Also compute $\text{var}(X)$ and $\text{var}(Y)$.

(ii) Let X be a random variable having Poisson distribution with parameter λ. Show that the moment generating function (mgf) of $Z = \dfrac{X-\lambda}{\sqrt{\lambda}}$ converges to the mgf of the standard normal distribution when $\lambda \to \infty$. 6

(c) (i) Let p denotes the probability of getting a head when a given coin is tossed once. Suppose that the hypothesis $H_0 : p = 0.5$ is rejected in favour of $H_1 : p = 0.6$ if 10 trials result in 7 or more heads. Calculate the probabilities of type I and type II errors. 6

(ii) If X is a continuous random variable, prove that the first absolute moment of X is minimum when taken about the median. 6

(d) (i) A random variable X has a density function $f(x)$ given by 6

$$f(x) = e^{-x}\ ,\quad x \ge 0$$
$$= 0\ ,\quad \text{elsewhere}$$

Show that Tchebycheff's inequality gives $P(|X-1| \ge 2) \le \frac{1}{4}$ and show that the actual probability is e^{-3}.

(ii) The integers x and y are chosen at random with replacement from the nine integers 1, 2, 3, ... , 8, 9. Find the probability that $|x^2 - y^2|$ is divisible by 2. 4

(iii) State Central limit theorem for independent and identically distributed (i.i.d) random variables with finite variance. 2

DSE1B
DIFFERENTIAL GEOMETRY
GROUP-A

1. Answer any ***four*** questions from the following: $3\times4=12$

(a) Define unit speed curve. Show that the curve
$$\gamma(t)=\left(\tfrac{1}{3}(1-t)^{3/2},\ \tfrac{1}{3}(1+t)^{3/2},\ \tfrac{t}{\sqrt{2}}\right) \text{ is unit speed regular.}$$

(b) Define orientable surface with an example.

(c) Define atlas of a surface. Write down an atlas of unit sphere.

(d) Find the arc length of the curve $\gamma(t)=(t,\ \cosh t)$ starting at the point (0, 1).

(e) Define the reparametrization of a curve. Find the reparametrization of the curve. 1+2
$$\gamma(t)=\left(\frac{2\cos t}{1+\sin t},\ 1+\sin t\right) \quad \text{for } -\frac{\pi}{2}<t<\frac{\pi}{2}$$

GROUP-B

2. Answer any ***four*** questions from the following: $6\times4=24$

(a) Find the curvature of the curve $\gamma(t)=(t-\cosh t\ \sinh t,\ 2\sinh t)$, $t>0$

(b) Calculate the first fundamental form of the surface $\sigma(u,v)=(\cosh u\cos v,\ \sinh u\sin v,\ u)$

(c) Find the evolute of the ellipse $\gamma(t)=(a\cos t,\ b\sin t)$, where $a>b>0$ are constants.

(d) Prove that a diffeomorphism $f:S_1\to S_2$ is an isometry if and only if, for any surface patch σ_1 of S_1, the patches σ_1 and $f\cdot\sigma_1$ of S_1 and S_2 respectively have the same first fundamental form.

(e) Prove that a parametrized curve has a unit-speed reparametrisation if and only if it is regular.

(f) For the surface, $\sigma(u,v)=\left(\frac{u+v}{2},\ \frac{v-u}{2},\ uv\right)$,

(i) Find the asymptotic curve on it.

(ii) Calculate the normal curvature of the curve $\gamma(t)=(t^2,\ 0,\ t^4)$ on the above surface.

GROUP-C

3. Answer any ***two*** questions from the following: $12\times2=24$

(a) State Frenet-Serret equations. Compute k, τ, t, n and b for the curve 1+10+1
$$\gamma(t)=\left(\tfrac{4}{5}\cos t,\ 1-\sin t,\ -\tfrac{3}{5}\cos t\right)$$
Also verify the Frenet-Serret equations.

(b) (i) State Gauss-Bonnet theorem for simple closed curve. 2

(ii) Define minimal surface. Prove that the surface 1+9
$$\sigma(u,v)=(\cosh u\cos v,\ \cosh u\sin v,\ u) \text{ is a minimal surface.}$$

(c) (i) Define principal curvature and principal tangent vector. Prove that if k_1 and k_2 be the principal curvatures at a point of a surface then k_1 and k_2 are both reals. 2+4

(ii) Find the principal curvature and corresponding principal tangent vector for the circular cylinder of radius 1 and axis z-axis represented by $\gamma(t)=(\cos v,\ \sin v,\ u)$ 6

(d) (i) Define Gaussian curvature at a point on a surface. 1

(ii) Prove that Gaussian-curvature $K=\dfrac{LN-M^2}{EG-F^2}$, where E, F, G and L, M, N are respectively first and second fundamental coefficients. 3

(iii) On the basis of the value of K, find the nature of the point at which K defined. 2

(iv) Find the Gaussian curvature of the surface $\sigma(u,v)=(u-v,\ u+v,\ u^2+v^2)$ at (5, 0, 1). 6

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