



‘সমানো মন্ত্র: সমিতি: সমানী’

UNIVERSITY OF NORTH BENGAL
B.Sc. Honours 3rd Semester Examination, 2023

CC5-MATHEMATICS

THEORY OF REAL FUNCTIONS AND INTRODUCTION TO METRIC SPACE

(REVISED SYLLABUS 2023 / OLD SYLLABUS 2018)

Time Allotted: 2 Hours

Full Marks: 60

The figures in the margin indicate full marks.

GROUP-A

1. Answer any **four** questions: 3×4 = 12
- (a) Evaluate $\lim_{x \rightarrow 3} \left([x] - \left[\frac{x}{3} \right] \right)$. 3
- (b) If a function $f : [0, 1] \rightarrow \mathbb{R}$ is continuous on $[0, 1]$ and f assumes only rational values on $[0, 1]$, then prove that f is a constant. 3
- (c) Discuss the applicability of M.V.T. for $f(x) = |x|$ in $[-1, 1]$ and $[0, 1]$. 3
- (d) Is the function $f(x) = \frac{x}{x+1}$ uniformly continuous for $x \in [0, 2]$? Justify your answer. 3
- (e) Prove that the intersection of two open sets in a metric space is open. 3
- (f) For a subset A of a metric space (X, d) , show that $\text{Int}(A) = X - \text{cl}(X - A)$. 3

GROUP-B

2. Answer any **four** questions: 6×4 = 24
- (a) If $f : [a, b] \rightarrow \mathbb{R}$ is continuous, then show that f attains its supremum and infimum on $[a, b]$. 6
- (b) Suppose $f(x)$ is a function satisfying the conditions: 6
- (i) $f(0) = 2, f(1) = 1$
- (ii) f has a minimum value at $x = \frac{1}{2}$
- (iii) $f'(x) = 2ax + b$ for all x .
- Determine the constants a, b and the function $f(x)$.
- (c) State and prove Rolle's Theorem. 6

- (d) If $f : (a, b) \rightarrow \mathbb{R}$ is continuous, then prove that f is uniformly continuous on (a, b) iff $\lim_{x \rightarrow a^+} f(x)$ and $\lim_{x \rightarrow b^-} f(x)$ exists finitely. 6
- (e) For a metric space (X, d) , show that $d^*(x, y) = \frac{d(x, y)}{1 + d(x, y)}, \forall x, y \in X$ is a bounded metric on X . 6
- (f) If $\{x_n\}$ and $\{y_n\}$ are two sequences in a metric space (X, d) such that $x_n \rightarrow x$ and $y_n \rightarrow y$, then prove that $d(x_n, y_n) \rightarrow d(x, y)$. 6

GROUP-C

3. Answer any *two* questions: 12×2 = 24
- (a) (i) Suppose $g : D \rightarrow \mathbb{R}$ be a function on $D \subseteq \mathbb{R}$ and $(c, \infty) \subset D$ for some $c \in \mathbb{R}$. 6
 Prove that $\lim_{x \rightarrow \infty} g(x) = l, l \in \mathbb{R}$ iff $\lim_{x \rightarrow 0^+} g\left(\frac{1}{x}\right) = l$.
- (ii) Using Cauchy's principle prove that $\lim_{x \rightarrow 0} \cos \frac{1}{x}$ does not exist. 6
- (b) (i) A function $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ and f is continuous at 0. Prove that f is continuous at every $c \in \mathbb{R}$. Also deduce that $f(x) = kx$ for all $x \in \mathbb{R}$ and for some $k \in \mathbb{R}$. 6
- (ii) A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined by $f(x) = x, x \in \mathbb{Q}; f(x) = 0, x \in \mathbb{R} \setminus \mathbb{Q}$. Show that f is continuous at 0 and f has a discontinuity of the 2nd kind at every other point in \mathbb{R} . 6
- (c) (i) Suppose a function $f : [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$ and $f'(x) \neq 0$ on (a, b) . If $f(a)$ and $f(b)$ are of opposite signs, show that there is a unique $x_0 \in (a, b)$ such that $f(x_0) = 0$. 6
- (ii) Find the global maximum and the global minimum of the function f on \mathbb{R} , where $f(x) = \frac{x^2 - 2x + 4}{x^2 + 2x + 4}, x \in \mathbb{R}$. 6
- (d) (i) Consider the space of all sequences of complex numbers and define the function d on X by 6

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{|x_n - y_n|}{(1 + |x_n - y_n|)}, \forall x = \{x_n\}, y = \{y_n\} \in X$$
 Show that (X, d) is a metric space.
- (ii) Suppose X be the set of all continuous real valued functions defined on $[0, 1]$ and $d(x, y) = \int_0^1 |x(t) - y(t)| dt, x, y \in X$. Show that (X, d) is not complete. 6

