'समानो मन्त्रः समितिः समानी'

UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 5th Semester Examination, 2023

CC11-MATHEMATICS

GROUP THEORY-II

(REVISED SYLLABUS 2023 / OLD SYLLABUS 2018)

Time Allotted: 2 Hours | Full Marks: 60

The figures in the margin indicate full marks.

GROUP-A

1. Answer any ***four*** questions from the following: $3\times4 = 12$

(a) Show that the characteristic subgroup of a group is normal. 3

(b) Find the number of inner automorphisms of the group S_3. 3

(c) Find the number of Sylow 2-subgroups of S_4 and A_4 . 3

(d) Find the number of non-isomorphic abelian groups of order $(2017)^3$. 3

(e) Find the conjugacy classes of the group D_3. 3

(f) Prove that the additive group $\mathbb{Z}\times\mathbb{Z}$ is not cyclic. 3

GROUP-B

Answer any *four* questions from the following $6\times4 = 24$

2. (a) Prove that a commutative group G is simple if and only if $G\cong\mathbb{Z}_p$, for some prime number p. 3

(b) Let G be an infinite cyclic group. Prove that $\mathrm{Aut}(G)\cong\mathbb{Z}_2$. 3

3. Let H be a subgroup of a group G. Consider a mapping $\sigma: H\times G\to G$, defined by $\sigma(h, g) = gh^{-1}$ for all $(h, g)\in H\times G$.

(a) Prove that this mapping defines an action of H on G. 4

(b) Find $\mathrm{Orb}(g)$ and $\mathrm{Stab}(g)$, where $g\in G$. Here $\mathrm{Orb}(g)$ denotes the orbit of 'g' and $\mathrm{Stab}(g)$ denotes the stabilizer of 'g' under this action. 2

4. State and prove Sylow's second theorem. 6

5. (a) Prove that there is no simple group of order 300. 4

(b) State the fundamental theorem of finite abelian group. 2

6. (a) Find the number of elements of order 5 in $\mathbb{Z}_{15} \times \mathbb{Z}_5$. 4

(b) Write the class equation of S_4. 2

7. Prove that direct product of two finite cyclic groups is cyclic if and only if orders of the cyclic groups are relatively prime. 6

GROUP-C

Answer any *two* questions from the following $12\times 2 = 24$

8. (a) Let N be a normal subgroup of a group G. Prove that G/N is abelian if and only if $[G, G]$ is a subgroup of N. Here $[G, G]$ denotes the commutator subgroup of G. 4

(b) Find $[A_4, A_4]$ and $[S_3, S_3]$. 4+4

9. (a) Show that the converse of Lagrange's theorem for finite abelian group is not true, in general. 4

(b) Prove that the center of a p-group is nontrivial. 4

(c) Show that every non-cyclic group of order 21 contains only 14 elements of order 3. 4

10.(a) Define automorphism of a group G. Prove that set of all automorphisms of a group G forms a group under function composition. If C_n be a cyclic group of order n prove that $\text{Aut}(C_n) \cong \mathbb{Z}_n^X$, an abelian group of order $\phi(n)$. 1+2+5

(b) Let G be a finite group and p be a prime integer. If p divides $|G|$ then prove that G has an element of order p. 4

11.(a) Prove that every group is isomorphic to some subgroup of the group S_A of all permutations of some set A. Using this result, prove that if G be a group and H be a subgroup of G of index n, then there exists a homomorphism ϕ from G into S_n such that $\ker\phi \subseteq H$. 4+4

(b) Prove that if a group G acts on itself by conjugation, then for each $a \in G$, $\text{Stab}(a) = Z_a$. Here Z_a denotes the centralizer of 'a'. 4

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