10. Derive the heat conduction equation. 6

11. Find the integral surface of $pq = xy$ which passes through the curve $z = x$, $y = 0$; by using the method of characteristics. 6

12. A particle is projected with a velocity V from the cusp of an inverted cycloid down the arc. Show that the time of reaching the vertex is $2\sqrt{\frac{a}{g}}\tan^{-1}\left(\frac{\sqrt{4ag}}{V}\right)$. 6

GROUP-C

Answer any *two* questions 12×2 = 24

13.(a) Determine the characteristics of the equation $z = p^2 - q^2$ and find the integral surface which passes through the parabola $4z + x^2 = 0,\ y = 0$. 6

(b) Show that the equations $xp - yq = x$, $x^2 p + q = xz$ are compatible and find their solution. 6

14.(a) Eliminating the arbitrary functions $f(x)$ and $g(y)$ from $z = yf(x) + xg(y)$, obtain the P.D.E. $xys = px + qy - z$. 6

(b) Deduce D'Alembert's formula of the Cauchy problem 6

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}; -\infty < x < \infty,\ t > 0 \text{ subject to the conditions:}$$

$$u(x, 0) = f(x),\ u_t(x, 0) = g(x) \text{ for } -\infty < t < \infty.$$

15.(a) Solve by Lagrange's method 4

$$py + qx = xyz^2(x^2 - y^2).$$

(b) State Cauchy-Kowalevski theorem and prove it for the following problem: 2+6

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = F(x, t),\ 0 < x < l,\ t > 0$$

subject to the conditions

$$u(x, 0) = f(x),\ u_t(x, 0) = g(x) \text{ for } 0 \le x < l \text{ and } u(0, t) = u(l, t) = 0,\ t \ge 0.$$

16.(a) Find the temperature distribution in a rod of length l. The faces are insulted and the initial temperature distribution is given by $x(l - x)$. 5

(b) Give an example of a quasi-linear P.D.E. Discuss the method of characteristic to solve the following quasilinear P.D.E. 1+4+2

$$Pp + Qq = R$$

and also show that the Lagrange's auxiliary equation is given by

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

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UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 6th Semester Examination, 2023

CC14-MATHEMATICS

PARTIAL DIFFERENTIAL EQUATIONS AND APPLICATIONS

Time Allotted: 2 Hours | Full Marks: 60

The figures in the margin indicate full marks.
Symbols have their usual meaning.

GROUP-A

Answer any *four* questions $3\times 4 = 12$

1. Determine the region in the xy-plane in which the P.D.E. $(1-x^2)u_{xx} = u_{yy}$ is hyperbolic. 3

2. Find the nature of the P.D.E. and find its characteristic variables: 3

$$u_{xx} + 2u_{xy} + 4u_{yy} + 2u_x + 3u_y = 0.$$

3. Solve $z = px + qy + p^2 + q^2$. 3

4. Define Cauchy problem for one dimensional wave equation. 3

5. Reduce the equation $u_{xx} + x^2 u_{yy} = 0$ to canonical form. 3

6. Show that the pedal equation of a central orbit is given by $\dfrac{h^2}{p^3}\dfrac{dp}{dr} = F$. 3

GROUP-B

Answer any *four* questions $6\times 4 = 24$

7. Solve $u_t = c^2 u_{xx}$; $u(0, t) = 0 = u(l, t)$ for all t and $u(x, 0) = f(x)$ for all $x \in [0, l]$. 6

8. By using D'Alembert's principle, solve the following P.D.E. 6

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2};\ -\infty < x < \infty$$

$$u(x, 0) = \sin x;\ \frac{\partial u}{\partial t}(x, 0) = 1.$$

9. Use method of separation of variables to solve 6

$$3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0,$$

$$u(x, 0) = 4e^{-x}.$$

(b) Let V be an inner product space and S_1 and S_2 be two subsets of V. Then prove that $S_1 \subseteq S_2 \Rightarrow S_1^{\perp} \subseteq S_2^{\perp}$. 3

9. (a) Let F be a field and $p(x) \in F[x]$. Then prove that $\langle p(x) \rangle$ is a maximal ideal in $F[x]$ iff $p(x)$ is irreducible over F. 4

(b) Prove that 2 and 5 are not irreducible elements in $\mathbb{Z}[i]$. 2

10.(a) Prove that in an integral domain, two elements a and b are associates iff $\langle a \rangle = \langle b \rangle$. 3

(b) Show that $1+2i$ and $3+5i$ are prime to each other in $\mathbb{Z}[i]$. 3

11.(a) In $\mathbb{R}^3$, with standard inner product, let P be the subspace span$\{(1, 1, 0), (0, 1, 1)\}$. Find $P^{\perp}$. 3

(b) Let V be a finite dimensional Euclidean space. Then prove that a linear mapping $T: V \to V$ is orthogonal iff T maps an orthonormal basis to another orthonormal basis. 3

12.(a) Prove that the set of all normal operators is a closed subset of $L(H, H)$ which contains the set of all self-disjoint operators. 4

(b) Suppose $A \in L(H, H)$. Then prove that $\langle Ax, x \rangle = 0$ for all $x \in H$ iff $A = 0$. 2

GROUP-C

Answer any *two* questions from the following $12 \times 2 = 24$

13.(a) Let $A: H \to H$ is a continuous linear operator, where H is a Hilbert space. Prove that A^* is a continuous linear operator with $\| A^* \| = \| A \|$. 5

(b) Prove that the dual space of an n dimensional vector space is n dimensional. 4

(c) Find the minimal polynomial of the matrix 3

$$A = \begin{pmatrix} -3 & 2 \\ 0 & -3 \end{pmatrix}$$

14.(a) Let T be an linear operator on $V = \mathbb{R}^2$ defined by $T(a, b) = (2a - 2b, -2a + 5b)$ for all $(a, b) \in \mathbb{R}^2$. Determine whether T is normal, self-adjoint or neither. Produce an orthonormal basis of eigenvectors of T for V. 6

(b) Let $V = P_1(\mathbb{R})$ and $W = \mathbb{R}^2$ with respective standard ordered bases β and γ. Define $T: V \to W$ by $T(p(x)) = (p(0) - 2p(1),\ p(0) + p'(0))$; where $p'(x)$ denotes the derivative of $p(x)$. Then compute $[T']_{\gamma^*}^{\beta^*}$ and $[T]_{\beta}^{\gamma}$. 6

15.(a) Apply Gram-Schmidt process to the subset $S=\{1, x, x^2\}$ of the inner product space $V=P_2(\mathbb{R})$ with inner product 4

$$\langle f, g\rangle=\int_0^1 f(t)g(t)\,dt$$

to obtain an orthonormal basis β for $\text{span}(S)$.

(b) For two subspaces W_1 and W_2 of a finite dimensional vector space V, prove that $(W_1+W_2)^0=W_1^0 \cap W_2^0$. 3

(c) Let T be a linear operator on a finite dimensional vector space V and let $f(t)$ be the characteristic polynomial of T. Then prove that $f(T)=T_0$, the zero transformation. 5

16.(a) Show that the following polynomials are irreducible: 6

(i) $x^3-[9]$ over $\mathbb{Z}_{11}$.

(ii) x^4+2x+2 over $\mathbb{Q}$.

(iii) x^6+x^3+1 over $\mathbb{Q}$.

(b) Let R be the ring $\mathbb{Z}\times\mathbb{Z}$. Show that the linear equation $(5, 0)x+(20, 0)=(0, 0)$ has infinitely many roots in R. 3

(c) In $\mathbb{Z}_7[x]$, factorize $f(x)=x^3+[1]$ into linear factors. 3

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UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 6th Semester Examination, 2023

CC13-MATHEMATICS

RING THEORY AND LINEAR ALGEBRA-II

Time Allotted: 2 Hours — Full Marks: 60

The figures in the margin indicate full marks.

GROUP-A

Answer any *four* questions from the following — $3\times 4 = 12$

1. Show that the polynomial $3x^5+15x^4-20x^3+10x+20$ is irreducible over $\mathbb{Q}$.

2. Suppose that a, b are two elements in an integral domain, $b\neq 0$ and a is not a unit. Show that $\langle ab\rangle$ is contained in $\langle b\rangle$.

3. Let $V=C[0,1]$ and define $\langle f, g\rangle=\int_0^{1/2} f(t)g(t)\,dt$. Is this an inner product on V?

4. Prove that the ideal $\langle x^2+1\rangle$ is prime in $\mathbb{Z}[x]$ but not maximal in $\mathbb{Z}[x]$.

5. Let $V=P_1(\mathbb{R})$ and for $p(x)\in V$, define $f_1, f_2\in V^*$, by $f_1(p(x))=\int_0^1 p(t)\,dt$ and $f_2(p(x))=\int_0^2 p(t)\,dt$. Prove that $\{f_1, f_2\}$ is a basis for V^*.

6. Prove that $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ is not diagonalizable.

GROUP-B

Answer any *four* questions from the following — $6\times 4 = 24$

7. (a) Prove that every Euclidean Domain is a PID. 4

   (b) Prove that in an integral domain, associates of every irreducible element are also irreducible. 2

8. (a) Let T be a linear operator on a finite dimensional vector space V and let W be a T-invariant subspace of V. Then prove that the characteristic polynomial of T_W divides the characteristic polynomial of T. 3