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UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 4th Semester Examination, 2023

CC9-MATHEMATICS

RING THEORY AND LINEAR ALGEBRA-I

(REVISED SYLLABUS 2023 / OLD SYLLABUS 2018)

Time Allotted: 2 Hours | Full Marks: 60

The figures in the margin indicate full marks.

GROUP-A

Answer any *four* questions from the following $3\times4 = 12$

1. Let F be a field, then show that the groups $(F\setminus\{0\}, \cdot)$ and $(F, +)$ cannot be isomorphic. 3

2. Find a basis and dimension of the subspace S of the vector space $M_2(\mathbb{R})$ over $\mathbb{R}$, where $S=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in M_2(\mathbb{R}) : a+b=0\right\}$. 3

3. Find a generator for each of the ideals $4\mathbb{Z}+10\mathbb{Z}$ and $8\mathbb{Z}\cap 12\mathbb{Z}$ of $\mathbb{Z}$. 3

4. Is the ring of matrices $\left\{\begin{pmatrix} a & b\\ 2b & a\end{pmatrix} : a, b\in\mathbb{R}\right\}$ a field? Justify your answer. 3

5. Let V be a vector space with $\{\alpha, \beta, \gamma\}$ as a basis. Prove that the set $\{\alpha+\beta+\gamma, \beta+\gamma, \gamma\}$ is also a basis of V. 3

6. Find a linear mapping $T:\mathbb{R}^3\rightarrow\mathbb{R}^3$ such that 3
$$\operatorname{Im}T=\{(x, y, z)\in\mathbb{R}^3 : x+y-z=0\}.$$

GROUP-B

Answer any *four* questions from the following $6\times4 = 24$

7. The matrix of a linear mapping $T:\mathbb{R}^3\rightarrow\mathbb{R}^3$ relative to the ordered basis $\{(-1, 1, 1), (1, -1, 1), (1, 1, -1)\}$ of $\mathbb{R}^3$ is $\begin{pmatrix} 1 & 2 & 2\\ 2 & 1 & 3\\ 3 & 3 & 1\end{pmatrix}$. Find T. Also find the matrix of T relative to the standard basis of $\mathbb{R}^3$. 3+3

8. Find all the units in the ring $\mathbb{Z}_{10}$. Prove that these units form a cyclic group under multiplication. 2+4

9. State and prove third isomorphism theorem on rings. 2+4

10. Determine the linear mapping $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ which maps the basis vectors $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ of $\mathbb{R}^3$ to the vectors $(1, 1), (2, 3), (3, 2)$ respectively. Prove that T is onto but not one-one. 3+3

11.(a) Prove that every Boolean ring is commutative. Is the converse true? Justify your answer. A ring is Boolean if its every element is idempotent. 3

(b) Let S and T be two ideals of a ring R. Prove that $S \cup T$ is an ideal of R iff either $S \subseteq T$ or $T \subseteq S$. 3

12.(a) In $\mathbb{R}^2$, consider $\alpha = (3, 1)$ and $\beta = (2, -1)$. Determine $L\{\alpha, \beta\}$ and show that $L\{\alpha, \beta\} = \mathbb{R}^2$. 3

(b) Prove that the set S of all 2×2 symmetric matrices with real entries is a subspace of $M_2(\mathbb{R})$. 3

GROUP-C

Answer any *two* questions from the following $12\times2 = 24$

13.(a) Prove that in a commutative ring R with identity, a proper ideal P of R is a prime ideal iff R/P is an integral domain. Use this result to prove that $\langle x \rangle$ is a prime ideal of $\mathbb{Z}[x]$. 4+2

(b) Give an example of each of the following: 2+2

(i) An infinite ring R (which is not a field) with char $R = 2$.

(ii) An infinite field F with char $F = 3$.

Here char S denotes the characteristic of the ring S.

(c) Prove that the characteristic of a Boolean ring is 2. 2

14.(a) Consider the subring R of $M_2(\mathbb{Z})$; where $R = \left\{\begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in \mathbb{Z}\right\}$. 2+2+2+2

Let $\phi : R \longrightarrow \mathbb{Z}$ be a map, defined by $\phi\left(\begin{pmatrix} a & b \\ b & a \end{pmatrix}\right) = a - b$ for all $\begin{pmatrix} a & b \\ b & a \end{pmatrix} \in R$.

(i) Show that ϕ is a homomorphism.

(ii) Determine $\ker\phi$.

(iii) Show that $R/\ker\phi$ is isomorphic to $\mathbb{Z}$.

(iv) Is $\ker\phi$ a prime ideal of R? Justify.

(b) Let $(R, +, \cdot)$ be a ring where $(R, +)$ is a cyclic group. Prove that R is a commutative ring. Use this result to prove that the rings of order 2, 3, 5, 6, 7 are commutative rings. 2+2

15.(a) Show that T is non-singular and determine T^{-1} where $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is a linear mapping defined by: 3+3

$$T(x, y, z) = (x - y, x + 2y, y + 3z) \text{ for all } (x, y, z) \in \mathbb{R}^3.$$

(b) Suppose a linear mapping $T: V \to W$ maps the ordered basis $\{\alpha_1, \alpha_2, \alpha_3\}$ of $V(\mathbb{R})$ as 4

$$T(\alpha_1) = \beta_1,\ T(\alpha_2) = \beta_1 + \beta_2,\ T(\alpha_3) = \beta_1 + \beta_2 + \beta_3$$

where $\{\beta_1, \beta_2, \beta_3\}$ is an ordered basis of $W(\mathbb{R})$. Find the matrix of T^{-1} relative to the same chosen ordered bases.

(c) Prove that the vector space $\mathbb{R}$ over $\mathbb{Q}$ is infinite dimensional. 2

16.(a) Let $W_1 = L\{(1, -2, 1), (2, 3, 5)\}$ and $W_2 = L\{(1, -2, 0), (3, -3, 0)\}$ then show that W_1 and W_2 are subspaces of $\mathbb{R}^3$. Determine $\dim W_1$, $\dim W_2$ and $\dim(W_1 + W_2)$. 4+4

(b) Find the dimension of $\ker T$ where $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is the linear transformation, given by $T(x, y, z) = (x + z, y + z)$ for all $(x, y, z) \in \mathbb{R}^3$. 4

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UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 4th Semester Examination, 2023

CC10-MATHEMATICS

METRIC SPACE AND COMPLEX THEORY

(REVISED SYLLABUS 2023 / OLD SYLLABUS 2018)

Time Allotted: 2 Hours | Full Marks: 60

The figures in the margin indicate full marks.

GROUP-A

1. Answer any ***four*** questions from the following: $3\times4=12$

(a) Show that the function $f: X \to Y$ is uniformly continuous, where X is discrete metric space and Y is any metric space.

(b) Give an example with proper justification of a set which is bounded but not totally bounded.

(c) Let X be a set and $|X|\geq 2$ with the discrete metric. Show that X is not connected.

(d) If f be an analytic function on a region $G(\subset \mathbb{C})$ such that $\operatorname{Im} f=0$, then show that f is constant.

(e) Show that every totally bounded metric space is separable.

(f) Find the Laurent series expansion of the function $\dfrac{7z-2}{z(z-2)(z+1)}$ in the domains $|z|>2$ and $1<|z|<2$ respectively. $1\frac{1}{2}+1\frac{1}{2}$

GROUP-B

Answer any *four* questions from the following $6\times4=24$

2. Show that the map $f:[0,1]\to[0,1]$ given by $f(x)=x-\dfrac{x^2}{2}$, is a weak contraction map but not contraction map. Also find its fixed points if exists. 6

3. Establish Cauchy-Riemann equations in the polar form for a function $f(z)$. 6

4. For any non-empty A of a metric space (X, d), show that the function $f: X\longrightarrow \mathbb{R}$ given by $f(x)=d(x, A)$; $x\in X$, is uniformly continuous. 6

5. State and prove the sufficient conditions for differentiability of a complex valued function $f(z)$ of a complex variable. 6

6. Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function in a region G. Verify whether the functions $\overline{f(z)}$, $f(\bar{z})$, $\overline{f(\bar{z})}$ are analytic or not in G. 2+2+2

7. (a) Let $f(z) = \frac{1}{z^2}$ and Γ be the straight line joining the points i and $3+i$. Show that 3+3

$$\left| \int_\Gamma f(z)\,dz \right| \leq 3.$$

(b) Evaluate $\int_{|z|=2} \frac{1}{(z^2+1)} dz$.

GROUP-C

Answer any *two* questions from the following 12×2 = 24

8. (a) Prove that union of two compact subsets of the metric space (X, d) is also compact. 6

(b) Let (X, d) be a metric space with $x_0 \in X$. Let $f: X \to \mathbb{R}$ be defined by $f(x) = d(x, x_0)$. Prove that f is uniformly continuous on X. 6

9. (a) State and prove Cauchy-Goursat theorem. 8

(b) Find the value of $\int_\Gamma \frac{dz}{z-a}$, if 4

(i) a lies inside Γ, and

(ii) a lies outside Γ.

10. (a) If $f(z)$ is analytic within and on a simple closed rectifiable curve Γ and z_0 is any point inside Γ, then prove that $f'(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{(z-z_0)^2} dz$. 6

(b) Expand $f(z) = \cos z$ in Taylor Series about $z = \pi/4$ and determine the region of convergence of the series. 6

11. (a) Suppose that $f(z) = u(x, y) + iv(x, y)$ be an entire function such that $u_y - v_x = -2$ for all $z(= x+iy) \in \mathbb{C}$. Verify the function $f(z)$ is constant or not. 5

(b) Prove that every polynomial of degree n has exactly n (not necessarily distinct) zeros. 3

(c) Evaluate $\int_\Gamma \frac{\log z}{(z-1)^3} dz$, where Γ is the circle $|z-2| = 3/2$. 4

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UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 4th Semester Examination, 2023

CC8-MATHEMATICS

MULTIVARIATE CALCULUS

(REVISED SYLLABUS 2023 / OLD SYLLABUS 2018)

Time Allotted: 2 Hours Full Marks: 60

The figures in the margin indicate full marks.

GROUP-A

1. Answer any ***four*** questions from the following: $3\times4 = 12$

(a) Examine whether $f(x, y)=\begin{cases} xy & ; \quad (x, y)\neq(0, 0) \\ 0 & ; \text{ if } (x, y)=(0, 0)\end{cases}$ is continuous at the origin. 3

(b) State Euler's theorem for homogeneous function of two variables. 3

(c) Find the gradient of $\vec{r}+\frac{1}{\vec{r}}$ where $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$. 3

(d) Find the volume under the plane $z=8x+6y$ over the region $R=\{(x, y): 0\le x\le 1 \text{ and } 0\le y\le 2\}$. 3

(e) Show that for any vector $\vec{a}$, $\operatorname{curl}\vec{a}$ is a solenoidal vector. 3

(f) Find the equation of the tangent plane to the surface $z=x^2+y^2$ at the point (1, 2, 5). 3

GROUP-B

Answer any *four* questions from the following $6\times4 = 24$

2. Let $f(x, y)=\begin{cases} \dfrac{x^3-y^3}{x^2+y^2} & ; \quad x^2+y^2\neq 0 \\ 0 & ; \text{ if } x^2+y^2=0\end{cases}$. 6

Show that f is continuous at (0, 0) but not differentiable at (0, 0).

3. Find the maximum and minimum values of the function $3x+4y$ on the circle $x^2+y^2=1$. 6

4. Prove that $\iiint_V \overline{\nabla}\times\overline{B}\, dV=\iint_S \hat{n}\times\overline{B}\, dS$, where V is the volume bounded by a closed surface S and $\hat{n}$ is the positive outward drawn normal (unit) to S. 6

5. Prove that $r^n\vec{r}$ is irrotational. Find n when it is solenoidal vector. 6

6. Use Stoke's theorem to evaluate 6

$$\oint_S (\sin z\, dx - \cos x\, dy + \sin y\, dz)$$

where S is the boundary of the rectangle:

$0 \le x \le \pi$, $0 \le y \le 1$ and $z = 3$

7. Evaluate the integral $\iint_R e^{(x+y)/(x-y)}\, dx\, dy$, where R is the trapezoidal region with vertices (1, 0), (2, 0), (0, –2) and (0, –1). 6

GROUP-C

Answer any *two* from the following 12×2 = 24

8. (a) Let $f(x, y) = \begin{cases} x^2 \sin\frac{1}{x} + y^2 \cos\frac{1}{y} & ; x \ne 0,\ y \ne 0 \\ x^2 \sin\frac{1}{x} & ;\ x \ne 0 \\ y^2 \cos\frac{1}{y} & ;\ y \ne 0 \\ 0 & ;\ x = 0 = y \end{cases}$ 2+2+2

Prove that the both partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at (0, 0) but none of them is continuous at (0, 0). Also, examine the differentiability of $f(x, y)$ at (0, 0).

(b) Use Stoke's theorem, prove that 3+3

(i) $\operatorname{curl} \operatorname{grad} \phi = 0$, where ϕ is a scalar function.

(ii) $\operatorname{div} \operatorname{curl} \vec{F} = 0$, where F is a vector field.

9. (a) If E be the region bounded by the circle $x^2 + y^2 - 2ax - 2by = 0$, show that 6

$$\iint_E \sqrt{x(2a - x) + y(2b - y)}\, dx\, dy = \frac{2\pi}{3}(a^2 + b^2)^{3/2}$$

(b) If $f(x, y) = \begin{cases} xy\frac{x^2 - y^2}{x^2 + y^2}, & (x, y) \ne (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$ 6

Show that $f_{xy}(0, 0) \ne f_{yx}(0, 0)$.

10.(a) Find the maximum value of $f(x, y, z) = x^2 y^2 z^2$ subject to the subsidiary condition $x^2 + y^2 + z^2 = c^2$ (x, y, z are positive). 6

(b) Find $\vec{\nabla}(\vec{\nabla} \cdot \vec{A})$, when $\vec{A} = \frac{\vec{r}}{r}$. 6

11.(a) If $\vec{a}$ is a constant vector, then prove that $\operatorname{curl}(\vec{r} \times \vec{a}) = -2\vec{a}$, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. 6

(b) Show that the area bounded by a simple closed curve C is given by $\frac{1}{2}\oint_C (x\,dy - y\,dx)$. Hence find area of the ellipse $x = a\cos\theta$ and $y = a\sin\theta$. 6

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