UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 2nd Semester Examination, 2021

CC3-MATHEMATICS

REAL ANALYSIS

Full Marks: 60

ASSIGNMENT

The figures in the margin indicate full marks.
All symbols are of usual significance.

GROUP-A

Answer *all* questions 2×5 = 10

1. (a) Find the derived set of the set $A=(0,2)\cup(1,3)\cap Q$, where Q is the set of rational numbers. 2

(b) Find all limit points of the sequence $(\sin n)_{n\in\mathbb{N}}$ 2

(c) Find a bijection from Z^+ to $Z^+\times Z^+$ where Z^+ is the set of all positive integers. 2

(d) Construct a sequence $(r_n)_{n\in\mathbb{N}}$ of rational numbers that converges to a given real number r. 2

(e) Examine if for any $A\subset\mathbb{R}$, $\overline{A}=\{x\in A\,;\,\exists$ a sequence (x_n) in A so that $x_n\to x\}$. 2

GROUP-B

Answer *all* questions 10×3 = 30

2. (a) Prove that the series $\frac{1}{x+1}+\frac{x}{x+2}+\frac{x^2}{x+3}+\ldots\ldots\ (x>0)$ converges if $x<1$ and diverges if $x\geq 1$. 5

(b) If $\sum\limits_{n=1}^{\infty}a_n^2$ is convergent, prove that $\sum\limits_{n=1}^{\infty}\frac{a_n}{n}$ is also convergent ($a_n>0\quad\forall n\in\mathbb{N}$). 5

3. (a) Show that $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\cdots\cdots=\ln 2$. 5

(b) Find $\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)\cdots\cdots=?$ 5

4. (a) Show that finite union of compact subsets of $\mathbb{R}$ is compact. What about infinite union in this regard? 5+2

(b) Show that arbitrary intersection of compact subsets of $\mathbb{R}$ is compact. 3

GROUP-C

Answer *all* questions 5×2 = 10

5. Check if the family of all finite subsets of the set of natural numbers is countable. 5

6. Check if the family $\zeta = \left\{ \left(r_n - \frac{1}{2^{n+1}},\ r_n + \frac{1}{2^{n+1}} \right);\ n \in \mathbb{N} \right\}$ is an open cover of $\mathbb{R}$ where $(r_n)_{n \in \mathbb{N}}$ is a linear array of all rational numbers. 5

GROUP-D

Answer *all* questions 5×2 = 10

7. Let the sequence (x_n) of real numbers converges to the real number x and $p : \mathbb{N} \longrightarrow \mathbb{N}$ is a bijection. Check if $x_{p(n)} \to x$. 5

8. Let $f : D \longrightarrow \mathbb{R},\ D \subset \mathbb{R}$ be a continuous function and (x_n) be a sequence in D.

(a) Examine if $(f(x_n)) \to f(x)$ if $x_n \to x \in D$. 3

(b) Examine if $(f(x_n))$ is a Cauchy sequence if (x_n) is Cauchy. 2

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UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 2nd Semester Examination, 2021

CC4-MATHEMATICS

Full Marks: 60

ASSIGNMENT

The figures in the margin indicate full marks.
All symbols are of usual significance.

GROUP-A

1. Answer ***all*** the questions: 2×5=10

(a) Calculate $\lim\limits_{t\to 3} \vec{r}(t)$, where $\vec{r}(t)=\left(\dfrac{2t-4}{t+1}\right)\hat{i}+\left(\dfrac{t}{t^2+1}\right)\hat{j}+(4t-3)\hat{k}$.

(b) Examine whether the vector valued function $\vec{r}(t)=t^2\hat{i}+e^t\hat{j}+\dfrac{1}{t+3}\hat{k}$ is continuous at $t=-3$ or not.

(c) Find the angle between the normals to the following surfaces $y^2+z^2=9$ and $2(x^2-z^2)=3y$ at the point (2, 2, 1).

(d) Show that the integral $\int\limits_C y\,dx+x\,dy$ is independent of the path C joining the points $P(0, 1)$ and $Q(1, 2)$.

(e) Find the particular integral of the differential equation $\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+2y=xe^{-x}$.

GROUP-B

2. Answer ***all*** the questions: 10×3=30

(a) (i) Solve the differential equation $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+4y=e^x\cos x$. 5+5=10

(ii) Apply the method of variation of parameters to solve $\dfrac{d^2y}{dx^2}-y=\dfrac{2}{1+e^x}$.

(b) (i) Solve the Euler's equation $x^2\dfrac{d^2y}{dx^2}-9x\dfrac{dy}{dx}+25y=0$. 5+5=10

(ii) Solve: $\dfrac{dx}{dt}+\dfrac{dy}{dt}-2y=2\cos t-7\sin t$

$\dfrac{dx}{dt}-\dfrac{dy}{dt}+2x=4\cos t-3\sin t$

(c) (i) Evaluate the integral of $\vec{F} = (yz + zx)\vec{i} + xz\,\vec{j} + (xy + 2z)\vec{k}$ along the circle $x^2 + y^2 = 1,\ z = 1$ from (0, 1, 1) to (1, 0, 1). 5+5=10

(ii) Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = (x^2 - 3y^2)\vec{i} + (y^2 - 2x^2)\vec{j}$ s and C the closed curve in *xy* plane given by $x = 3\cos t$, $y = 2\sin t$, $0 \le t \le 2\pi$, C is described in the anti-clockwise sense.

GROUP-C

3. Answer ***all*** the questions: 5×2=10

(a) Solve $(D^2 - 2D + 4)y = (x + x^3)e^{2x}$ by method of undetermined coefficient.

(b) Apply Picard's method up to third approximation to solve

$$\frac{dy}{dx} = 3e^x + 2y \ ; \quad y(0) = 0 .$$

GROUP-D

4. Answer ***all*** the questions: 5×2=10

(a) Show that equation of the tangent line to the curve $x = t,\ y = t^2\ ,\ z = \frac{2}{3}t^3$ at the point $t = 1$ is $2(x-1) = (y-1) = z - \frac{2}{3}$.

(b) Solve:
$$\frac{dx}{dt} + 2x - 3y = t$$
$$\frac{dy}{dt} - 3x + 2y = e^{2t}$$

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UNIVERSITY OF NORTH BENGAL

B.Sc. Programme 2nd Semester Examination, 2021

DSC2-MATHEMATICS

ALGEBRA

Full Marks: 60

ASSIGNMENT

The questions are of equal value.
The figures in the margin indicate full marks.

GROUP-A

বিভাগ-ক

1. Answer ***all*** the questions from the following: 2×5=10

নিম্নলিখিত ***সবগুলি*** প্রশ্নের উত্তর দাওঃ

(a) Verify Cayley Hamilton theorem for the square matrix $\begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix}$.

$\begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix}$ বর্গ ম্যাট্রিক্সটির সাপেক্ষে Cayley Hamilton উপপাদ্যটি যাচাই কর।

(b) A relation ρ is defined on $\mathbb{Z}$ by $x\rho y$ iff $x^2 - y^2$ divisible by 5. Prove that ρ is an equivalence relation on $\mathbb{Z}$.

অখণ্ড সংখ্যাসমূহের সেট $\mathbb{Z}$-এর ওপর একটি সম্বন্ধ ρ -এর সংজ্ঞা এইরূপঃ

$$x\,\rho\,y \Leftrightarrow x^2 - y^2,\ 5 \text{ দ্বারা বিভাজ্য।}$$

দেখাও যে ρ একটি সমতুল্যতা সম্বন্ধ (equivalence relation)।

(c) Use Descartes' rule of signs to find the number of positive roots of the equation $x^6 + x^4 + x^2 + x + 3 = 0$.

'Descartes' rule of sign' (Descarte-এর চিহ্ন সংক্রান্ত সূত্রগুলো দ্বারা $x^6 + x^4 + x^2 + x + 3 = 0$ সমীকরণটির ধনাত্মক বীজগুলির সংখ্যা নির্ণয় কর।

(d) If $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, find the rank of $2A - A^2$.

যদি $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ হয়, তবে $2A - A^2$ -এর rank নির্ণয় কর।

(e) Show that the map $f:\ \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = x^{2020}$ is neither injective nor surjective, $\mathbb{R}$ being the set of reals.

$\mathbb{R}$ যদি বাস্তব সংখ্যার সেট হয় এবং $f:\ \mathbb{R} \longrightarrow \mathbb{R}$ অপেক্ষকটি $f(x) = x^{2020}$ দ্বারা সংজ্ঞাত, তাহলে প্রমাণ কর f অপেক্ষকটি ঐকৈক (injective) কিংবা উপরি (surjective) কোনোটাই নয়।

GROUP-B

বিভাগ-খ

Answer *all* the questions from the following 12×3=36

নিম্নলিখিত *সবগুলি* প্রশ্নের উত্তর দাও

2. (a) Prove that $\tan\left\{i\log\frac{a-ib}{a+ib}\right\}=\frac{2ab}{a^2-b^2}$. 4

প্রমাণ কর $\tan\left\{i\log\frac{a-ib}{a+ib}\right\}=\frac{2ab}{a^2-b^2}$ ।

(b) Reduce the equation $x^3-3x^2+12x+16=0$ to its standard form and then solve the equation by Cardan's method. 4

$x^3-3x^2+12x+16=0$ সমীকরণটিকে ইহার Standard form-এ রূপান্তরিত কর এবং Cardan পদ্ধতিতে উক্ত সমীকরণটিকে সমাধান কর।

(c) Prove that the roots of the equation $\frac{1}{x+a_1}+\frac{1}{x+a_2}+\cdots\cdots\cdots+\frac{1}{x+a_n}=\frac{1}{x}$ are all real, where $a_1, a_2, \cdots\cdots, a_n$ are all positive real numbers. 4

প্রমাণ কর যে $\frac{1}{x+a_1}+\frac{1}{x+a_2}+\cdots\cdots\cdots+\frac{1}{x+a_n}=\frac{1}{x}$ সমীকরণটির সকল বীজগুলি বাস্তব, যেখানে $a_1, a_2, \cdots\cdots, a_n$ সকলে ধনাত্মক বাস্তব সংখ্যা।

3. (a) Show that $19^{20}\equiv 1 \pmod{181}$. 3

প্রমাণ কর যে $19^{20}\equiv 1 \pmod{181}$ ।

(b) Using Euclidean Algorithm, find two integers u and v satisfying $1269u+297v=135$. 4

$1269u+297v=135$ সমীকরণকে সীদ্ধ করে এমন দুটি পূর্ণসংখ্যা u এবং v কে Euclidean algorithm-এর সাহায্যে নির্ণয় কর।

(c) Use Principle of Mathematical Induction to prove that for all $n\in\mathbb{N}$, 5

$$1.1!+2.2!+\cdots\cdots\cdots+n.n!=(n+1)!-1$$

গাণিতিক আবেশ নীতির (Principle of Mathematical Induction) দ্বারা দেখাও যে সকল $n\in\mathbb{N}$-এর জন্য $1.1!+2.2!+\cdots\cdots\cdots+n.n!=(n+1)!-1$.

4. (a) Use Cayley-Hamilton theorem to find A^{100}, where $A=\begin{bmatrix}1&0&0\\1&0&1\\0&1&0\end{bmatrix}$. 4

যদি $A=\begin{bmatrix}1&0&0\\1&0&1\\0&1&0\end{bmatrix}$ হয় তাহলে Cayley-Hamilton উপপাদ্যের সাহায্যে A^{100} এর মান নির্ণয় কর।

(b) Find the eigenvalue and the corresponding eigenvector of the matrix $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix}$. 5

$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix}$ ম্যাট্রিক্সটির আইগেনমান (Eigenvalue) এবং ইহার আইগেন ভেক্টর (eigenvector) গুলি নির্ণয় কর।

(c) Let λ_1 and λ_2 be two distinct eigen values of a real square matrix A. If u and v are eigen vectors of A corresponding to the eigen values λ_1 and λ_2 respectively, examine whether $u+v$ an eigen vector of A. 3

ধর λ_1 এবং λ_2 কোন একটি বাস্তব বর্গ ম্যাট্রিক্স A-এর দুটি পৃথক আইগেন মান। যদি u এবং v যথাক্রমে λ_1 এবং λ_2 আইগেন মানের আইগেন ভেক্টর হয় তবে $u+v$, A ম্যাট্রিক্সটির একটি আইগেন ভেক্টর হবে কিনা যাচাই কর।

GROUP-C

বিভাগ-গ

Answer *all* the questions from the following 7×2=14

নিম্নলিখিত *সবগুলি* প্রশ্নের উত্তর দাও

5\. (a) If a, b, c are three positive rational numbers then prove that 5

যদি a, b, c তিনটি ধনাত্মক মূলদ (Rational) সংখ্যা হয় তাহলে প্রমাণ করঃ

$$\left(\frac{a^2+b^2+c^2}{a+b+c}\right)^{a+b+c} \geq a^a b^b c^c \geq \left(\frac{a+b+c}{3}\right)^{a+b+c}$$

(b) If A, B are square matrices of same order over the field F, A is non-singular prove that the matrices B and ABA^{-1} have the same eigen values. 2

যদি A এবং B দুটি সমান ক্রম (order)-এর বর্গম্যাট্রিক্স যেখানে A হল non-singular, তাহলে প্রমাণ কর B এবং ABA^{-1} ম্যাট্রিক্সদ্বয়ের আইগেন মানগুলি একই হবে।

6\. Determine the conditions for which the system of equations 2+2+3=7

$$x+y+z=1$$
$$x+2y-z=b$$
$$5x+7y+az=b^2$$

admits of (i) only one solution (ii) no solution (iii) many solutions.

একটি মাত্র সমাধান (i) কোন্ সমাধান নেই (ii) একাধিক সমাধানের জন্য

$$x+y+z=1$$
$$x+2y-z=b$$
$$5x+7y+az=b^2$$

সমীকরণ তন্ত্রের (system of equations) শর্তগুলি উল্লেখ কর।

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UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 2nd Semester Examination, 2021

GE-MATHEMATICS

Full Marks: 60

ASSIGNMENT

The figures in the margin indicate full marks.
All symbols are of usual significance.

The question paper contains MATHGE-I, MATHGE-II, MATHGE-III, MATHGE-IV & MATHGE-V.
The candidates are required to answer any *one* from the *five* courses.
Candidates should mention it clearly on the Answer Book.

MATHGE-I

Cal. Geo and DE.

GROUP-A

1. Answer ***all*** the questions from the following: 2×5=10

(a) Find the asymptotes parallel to coordinate axes of the curves $y = \tan^{-1} x$.

(b) Evaluate: $\lim\limits_{x \to a} \left(2 - \frac{a}{x}\right)^{\tan\frac{\pi x}{2a}}$

(c) Solve: $x\,dy - y\,dx = \cos\frac{1}{x}\,dx$

(d) Evaluate: $\int_0^1 \frac{x^6}{\sqrt{1-x^2}}\,dx$

(e) Find the equation of a sphere whose great circle is $x^2 + y^2 + z^2 + 10y - 4z = 8$, $x + y + z = 3$.

GROUP-B

Answer *all* the questions from the following 12×3=36

2. (a) Find the reduction formula for $\int (\log x)^n x^m\,dx$ and hence find the value of $\int (\log x)^2 x\,dx$. 3

(b) Find the volume of the solid obtained by revolution the region bounded by $y = x$ and $y = x^2 - 2x$ about the line $y = 5$. 4

(c) Solve: $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$ 5

3. (a) Determine the point of inflection for $y = \sin^2 x$, $0 \le x \le 2\pi$. 3

(b) If $x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$ is the envelope of the lines $\frac{x}{a} + \frac{y}{b} = 1$, where a, b are variable parameter and c is a constant, then prove that $a^2 + b^2 = c^2$. 4

(c) A variable sphere passes through (0, 0, $\pm c$) and cuts the lines $y = x\tan\alpha$, $z = c$ and $y = -x\tan\alpha$, $z = -c$ in points P and P'. If $PP' = 2a$ (a is a constant), show that the centre of the sphere lies on the circle $x^2 + y^2 = (a^2 - c^2)\operatorname{cosec}^2 2\alpha$, $z = 0$. 5

4. (a) Find the value of a and b if the $\lim_{x\to\infty} \frac{a\sin^2 x + b\log\cos x}{x^4} = -\frac{1}{2}$. 3

(b) Solve: $\frac{dy}{dx} + 2xy = x^2 + y^2$ 4

(c) Reduce the canonical form $5x^2 - 20xy - 5y^2 - 16x + 8y - 7 = 0$. 5

GROUP-C

Answer *all* the questions from the following 7×2=14

5. (a) Find the 21st derivative of $y = \frac{x^3}{x^2 - 3x - 2}$. 3

(b) Trace the curve $x^2y^2 = (a+y)^2(a^2 - y^2)$. 4

6. (a) Solve: $(1+y^2)\,dx = (\tan^{-1} y - x)\,dy$ 3

(b) If $I_m = \frac{d^m}{dx^m}(x^m \log x)$, prove that $I_m = mI_{m-1} + (m-1)!$. Hence prove that $I_m = \left[\log x + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\cdots + \frac{1}{m}\right] m!$. 4

MATHGE-II

Algebra

GROUP-A

1. Answer ***all*** the questions from the following: 2×5=10

(a) Find the value of $(-i)^{1/4}$. 2

(b) Find the values of k for which the equation $x^4+4x^3-2x^2-12x+k=0$ has four real distinct roots. 2

(c) Let a, b, c be real numbers. Show that 2

$$(a+b-c)^2+(b+c-a)^2+(c+a-b)^2 \geq ab+bc+ca$$

(d) Check whether the function $f:\mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x)=x^2+x$, $\forall x\in\mathbb{R}$ is surjective or not. 2

(e) Find the eigenvalues and eigenvectors of the matrix $7I_7$. 2

GROUP-B

Answer *all* the questions from the following 12×3=36

2. (a) Let z_1, z_2 be two complex numbers such that z_1/z_2 is real. Prove that the points representing z_1 and z_2 in the complex plane are collinear with the origin. 3

(b) Let $z=\cos\theta+i\sin\theta$ and m be a positive integer. Show that 5

$$(1+z)^m+\left(1+\frac{1}{z}\right)^m=2^{m+1}\cos^m\frac{\theta}{2}\cos\frac{m\theta}{2}$$

(c) Show that $\tan 4\theta=\dfrac{4\tan\theta-4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}$. 4

3. (a) Apply Descartes' Rule of signs to find the nature of the roots of the equation $x^6-3x^2-2x+3=0$. 3

(b) If $ax\equiv ay\ (\text{mod } m)$ where a is prime to m then prove that $x\equiv y\ (\text{mod } m)$. 4

(c) Let α, β, γ be the roots of the equation $x^3+2x^2-3x+1=0$. Find the equations whose roots are $\alpha\beta-\gamma^2$, $\beta\gamma-\alpha^2$, $\gamma\alpha-\beta^2$ and deduce the condition that the roots of the given equation may be in geometric progression. 4+1

4. (a) Prove that the relation $\rho=\{(a,b): a\equiv b\ (\text{mod } 3)\ ;\ a, b\in\mathbb{Z}\}$ is an equivalence relation on the set of integers $\mathbb{Z}$. Also, find the sets $A=\{a:(a,1)\in\rho\}$, $B=\{a:(a,2)\in\rho\}$ and $C=\{a:(a,3)\in\rho\}$ where $a\in\mathbb{Z}$, show that $\mathbb{Z}=A\cup B\cup C$ and $A\cap B=B\cap C=C\cap A=\phi$. 3+4

(b) Reduce the following matrix: 4+1

$$\begin{bmatrix} 0 & 0 & 2 & 2 & 0 \\ 1 & 3 & 2 & 4 & 1 \\ 2 & 6 & 2 & 6 & 2 \\ 3 & 9 & 1 & 10 & 6 \end{bmatrix}$$

into row reduced echelon form and hence find its rank.

GROUP-C

Answer *all* the questions from the following 7×2=14

5. (a) Prove that $1!\cdot 3!\cdot 5!\cdot\cdots\cdots(2n-1)! > (n!)^n$. 3

(b) If $x_0, x_1, \ldots\ldots, x_{n-1}$ are the roots of $x^n - 1 = 0$, then prove that $(x_0 - x_1)(x_0 - x_2)\ldots..(x_0 - x_{n-1}) = n$. 4

6. (a) Use Cayley-Hamilton Theorem to find A^{2021}, where $A = \begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix}$. 2

(b) Determine the conditions for which the system of equations 5

$$x + y + z = b$$
$$2x + y + 3z = b + 1$$
$$5x + 2y + az = b^2$$

admits of (i) no solution, (ii) unique solution and (iii) many solutions.

MATHGE-III

Differential Equation and Vector Calculus

GROUP-A

1. Answer ***all*** the questions from the following: 2×5=10

(a) Show that $\sin 3x,\ \cos 3x$ is not a linearly independent solution of $y'' + 9y = 0$.

(b) Solve the equation: $\frac{dx}{dt} = -wy$ and $\frac{dy}{dt} = wx$.

(c) Find the particular integral of $\frac{d^2y}{dx^2} + y = \cos x$.

(d) If $\boldsymbol{r} = 3t\boldsymbol{i} + 3t^2\boldsymbol{j} + 2t^3\boldsymbol{k}$, then find $\frac{d\boldsymbol{r}}{dt} \times \frac{d^2\boldsymbol{r}}{dt^2}$.

(e) Given $\vec{r} = a\cos t\,\hat{i} + a\sin t\,\hat{j} + bt\,\hat{k}$. Show that $\left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right] = a^2 b$.

GROUP-B

Answer *all* the questions from the following 12×3=36

2. (a) Solve, using the method of variation parameters, the following differential equation: 6

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = \log x$$

(b) Evaluate the line integral $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ along the curve $C: x^2+y^2=1,\ z=2$ in the positive direction from $A(1, 0, 2)$ to $B(0, 1, 2)$ where 6

$$\boldsymbol{F}=(y+xz^2)\boldsymbol{i}+(z-y)\boldsymbol{j}+(xy-z)\boldsymbol{k}$$

3. (a) Solve, using the method of undetermined coefficients, the equation 6

$$(D^2-3D)y=x+e^x\sin x\ ,\quad \left(D\equiv\frac{d}{dx}\right)$$

(b) Evaluate $\left(\iint \vec{F}\cdot\hat{n}\right)ds$, where $\vec{F}=yz\hat{i}+x\hat{j}+z\hat{k}$ and S is the surface of the cylinder $x^2+y^2=4$ included in the first octant between $z=0$ and $z=2$. 6

4. (a) Solve the differential equation $\dfrac{d^2y}{dx^2}+a^2y=\sec ax$ with the symbolic operator D. 6

(b) If $\vec{F}(x,\ y,\ z)=r^n(\vec{\alpha}\times\vec{r})$, then prove that $\vec{F}$ is solenoidal, where $\vec{\alpha}$ is a constant vector, $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$ and $r=|\vec{r}|$. 6

GROUP-C

Answer *all* the questions from the following 7×2=14

5. Find the complete solution of the given differential equation 7

$$(D^3+1)y=e^{2x}\sin x+e^{\frac{x}{2}}\sin\left(\frac{\sqrt{3}x}{2}\right)$$

6. What is an irrotational vector? Find the constants *a*, *b*, *c* so that the vector 7

$$\vec{F}=(x+2y+az)\vec{i}+(bx-3y-z)\vec{j}+(4x+cy+2z)\vec{k}$$

is irrotational. Hence, find the potential function *V*.

MATHGE-IV

Group Theory

GROUP-A

1. Answer ***all*** the questions from the following: 2×5=10

(a) If $G=(\mathbb{R},\ +)$ and $H=(\mathbb{Z},\ +)$, then how many distinct left cosets of H in G are there?

(b) Let $H:=\{a+bi:\ a,\ b\in\mathbb{R},\ ab\geq 0\}$. Prove or disprove that H is a subgroup of $\mathbb{C}$ of all complex numbers under addition.

(c) Find an isomorphism from the group of integers $(\mathbb{Z},\ +)$ to the group of even integers $(2\mathbb{Z},\ +)$.

(d) "Every abelian group is cyclic" — Check whether the statement is true or false with proper justification.

(e) Check whether the set of rational numbers $\mathbb{Q}$ forms a group with respect to multiplication.

GROUP-B

Answer *all* the questions from the following 12×3=36

2. (a) Find the number of elements of order 5 in the group $(\mathbb{Z}_{30}, +)$. 5

(b) Give an example of a group G and it's subgroup H such that $[G:H]=2$. 5

(c) Let $(\mathbb{Z}, +)$ be the group of integers and $N=\{3n \mid n\in\mathbb{Z}\}$. Prove that N is a normal subgroup of $\mathbb{Z}$. 2

3. Let G be a group and $g\in G$. Define a map $\varphi_g : G\to G$ by $\varphi_g(x) := gxg^{-1}$ for all $x\in G$. Show that φ_g is an isomorphism from G onto itself. 6+6

Further, define $\mathcal{L}(G) := \{\varphi_g : g\in G\}$. Show that $\mathcal{L}(G)$ forms a group under composition of functions.

4. (a) If G is a cyclic group with only one generator, prove that either $o(G)=1$ or $o(G)=2$. 4

(b) Give an example of two subgroups H, K of a group which are not normal but HK is a subgroup. 4

(c) Suppose a group G has a subgroup of order n. Show that the intersection of all subgroups of G of order n is a normal subgroup of G. 4

GROUP-C

Answer *all* the questions from the following 7×2=14

5. (a) Prove that a cyclic group G of order n is isomorphic to the multiplicative group consisting of all n^{th} roots of unity. 5

(b) Show that for any group G, $G/\{e\}\cong G$. 2

6. (a) Let N be a normal subgroup of a group G. If N is cyclic, prove that every subgroup of N is also normal in G. 4

(b) Write down the cyclic decomposition of the following permutation as a product of disjoint cycles 3

$$\sigma=\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7 & 8\\ 4 & 5 & 6 & 1 & 3 & 2 & 8 & 7\end{pmatrix}$$

MATHGE-V

Numerical Methods

GROUP-A

1. Answer ***all*** the questions from the following: 2×5=10

(a) Given that $u=\frac{5xy^2}{z^3}$ find the relative error at $x=y=z=1$ when the errors in each of x, y, z is 0.001.

(b) Write the sufficient condition for convergence of Gauss-Seidel iteration method.

(c) The equation $x^2+ax+b=0$ has two real roots α and β. Show that $x_{k+1}=-(ax_k+b)/x_k$ is convergent near $x=\alpha$ if $|\alpha|>|\beta|$.

(d) Calculate $\sqrt{25.11}-\sqrt{25.1001}$, correct up to three significant figures.

(e) Show that $\Delta^n[ke^{ax}]=k(e^{ah}-1)^n e^{ax}$.

GROUP-B

Answer *all* the questions from the following 12×3=36

2. (a) Find the polynomial of degree three relevant to the following data by Lagrange's formula: 6

x	-1	0	1	2
$f(x)$	1	1	1	-3

(b) Evaluate $\int_2^3 \frac{dx}{1+2x}$ by Trapezoidal Rule taking $n=10$ and compare the result with the exact value of the integration. 6

3. (a) Show that the Bisection method converges linearly. 6

(b) Determine a, b and c such that the formula 6

$$\int_0^h f(x)\,dx = h\left[af(0)+bf\left(\frac{h}{3}\right)+cf(h)\right]$$

is exact for polynomial of as high order as possible and determine the order of truncation error.

4. (a) Use Picard's method to solve the differential equation $\frac{dy}{dx}=x^2+\frac{y}{2}$ at $x=0.5$, correct to two decimal places, given that $y=1$ when $x=0$. 6

(b) Solve the following system of linear equations by Gauss-Seidel method: 6

$$20x+5y-2z=14$$
$$3x+10y+z=17$$
$$x-4y+10z=23$$

GROUP-C

Answer *all* the questions from the following 7×2=14

5. Using modified Euler's method, find $y(4.4)$ by taking h = 0.2, from the following differential equation: 7

$$5x\frac{dy}{dx}+y^2-2=0\ ,\quad y(4)=1$$

6. If the third order differences of a function $f(x)$ are constants and 7

$$\int_{-1}^{1} f(x)\,dx = k\left[f(0)+f\left(\frac{1}{\sqrt{2}}\right)+f\left(-\frac{1}{\sqrt{2}}\right)\right],$$

then find the value of *k*.

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