UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 3rd Semester Examination, 2019

CC5-MATHEMATICS

Time Allotted: 2 Hours | Full Marks: 60

The figures in the margin indicate full marks
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

GROUP-A

1. Answer any ***four*** questions from the following: $3\times 4=12$

(a) If $f:[a,b]\to\mathbb{R}$ has derivative at $c\in(a,b)$ then prove that $\exists\,\delta>0$ and $M>0$ such that $|f(x)-f(c)|<M|x-c|$, $\forall x\in N_\delta(c)$.

(b) Let (X,d) be a metric space and $\delta(A\cup B)\le\delta(A)+\delta(B)+d(A,B)$, where δ designates the diameter of the set.

(c) Use sequential criterion of continuity to prove that the function f defined on $\mathbb{R}$ by,

$$f(x)=\begin{cases}1, & x\in Q\\ 0, & x\in\mathbb{R}|Q\end{cases}$$

is discontinuous at every point $c\in\mathbb{R}$.

(d) Prove that between any two real roots of the equation $e^x\sin x+1=0$ there is at least one root of $\tan x+1=0$.

(e) Show that $\lim_{x\to\infty} a^x\sin\frac{b}{a^x}=\begin{cases}0, & \text{if } 0<a<1\\ b, & \text{if } a>1\end{cases}$.

(f) Let $f:[-1,1]\to\mathbb{R}$ be defined by

$$f(x)=\begin{cases}0, & x\in[-1,0]\\ 1, & x\in[0,1]\end{cases}$$

Does there exist a function g such that $g'(x)=f(x)$, $x\in[-1,1]$.

GROUP-B

2. Answer any ***four*** questions from the following: $6\times 4=24$

(a) (i) If a function $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and injective on $[a,b]$, then f is strictly monotone on $[a,b]$. 4

(ii) A function $f:\mathbb{R}\longrightarrow\mathbb{R}$ satisfies the condition $|f(x)-f(y)|\le|x-y|^2$ for all $x,\ y\in\mathbb{R}$. Prove that f is a constant function on $\mathbb{R}$. 2

(b) (i) Show that if h be the height of a closed cylinder of given volume V and least surface area S, then the diameter is equal to h. 4

(ii) If $f(0)=f'(0)=0$ and $f''(x)$ exists in $0\le x\le h$, prove that $\exists\ c$ with $f(h)=\frac{1}{2}h^2f''(c)$, $0<c<h$. 2

(c) Show that $\lim\limits_{h\to 0}\theta=\dfrac{1}{n+1}$, where θ is given by, 6

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\dots\dots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{h^n}{n!}f^n(a+\theta h),$$

provided that f^{n+1} is continuous at a and $f^{n+1}(a)\neq 0$.

(d) Let ℓ_p be the set of all real sequences for which $\sum\limits_{i=1}^{\infty}|x_i|^p<\infty$ and a metric d in ℓ_p is defined by $d(x,\ y)=\left(\sum\limits_{i=1}^{\infty}|x_i-y_i|^p\right)^{1/p}$, $\forall\, x=\{x_i\}$ and $y=\{y_i\}\in\ell_p$. Then prove that the space $(\ell_p,\ d)$ is complete metric space. 6

(e) (i) Let $f:[a,\ b]\longrightarrow\mathbb{R}$ be differentiable at an interior point C of $[a,\ b]$. Let $\{\alpha_n\},\ \{\beta_n\}$ be two sequences satisfying $a<\alpha_n<C<\beta_n<b$ for all $n\in\mathbb{N}$, both converge to C. Prove that $\lim\limits_{n\to\infty}\dfrac{f(\beta_n)-f(\alpha_n)}{\beta_n-\alpha_n}=f'(C)$. 4

(ii) If $\lim\limits_{x\to a}f(x)=l\neq 0$, then prove that $\exists\ \delta>0$ such that $\frac{1}{2}|l|<|f(x)|<\frac{3}{2}|l|$, where $0<|x-a|<\delta$. 2

(f) The functions $f,\ \phi,\ f',\ \phi'$ are all continuous in $[a,\ b]$ and $f(x)\phi'(x)-f'(x)\phi(x)\neq 0,\ \forall\, x\in[a,\ b]$. Show that between any two roots of $f(x)=0$ in the interval lies one root of $\phi(x)=0$ and conversely. 6

GROUP-C

3. Answer any *two* questions from the following: $12\times 2=24$

(a) (i) Let $(Y,\ d')$ be a subspace of a metric space $(X,\ d)$. Prove that a set $A\subset Y$ is open in $(Y,\ d')$ if and only if $\exists$ an open set G in $(X,\ d)$ such that $A=G\cap Y$. 6

(ii) Let a function $f:[a,\ \infty)\longrightarrow\mathbb{R}$ be twice differentiable on $[a,\ \infty)$ and there exist positive real numbers A and B such that $|f(x)|\le A,\ |f''(x)|\le B$ for all $x\in[a,\ \infty)$. Prove that $|f'(x)|\le 2\sqrt{AB},\ \forall\, x\in[a,\ \infty)$. 3

(iii) Find the points of discontinuity of the function f defined by $f(x)=\lim_{n\to\infty}\left[\lim_{t\to 0}\frac{\sin^2(n!\pi x)}{\sin^2(n!\pi x)+t^2}\right]$, $x\in\mathbb{R}$. 3

(b) (i) Let a function f be continuous on an open bounded interval (a, b). Then f admits of a continuous extension to $\mathbb{R}$ if and only if f be uniformly continuous on (a, b). 6

(ii) If $f:[a, b]\longrightarrow\mathbb{R}$ be differentiable on $[a, b]$, then prove that the derived function f' cannot have a jump discontinuity on $[a, b]$. 6

(c) (i) If f is differentiable on $[0, 1]$. Show by Cauchy's Mean Value theorem that the equation $f(1)-f(0)=\frac{f'(x)}{2x}$ has at least one solution in $(0, 1)$. 5

(ii) Prove that two metrics d_1 and d_2 on a non-empty set X are equivalent if there exist two positive real numbers α, β such that for all $x, y\in X$, $\alpha d_1(x, y)\le d_2(x, y)\le \beta d_1(x, y)$. 7

(d) (i) Find Taylor's series expansion of $f(x)=(1+x)^m$, $x\in\mathbb{R}$ for different values of m. 6

(ii) State and prove Darboux's theorem. 6

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UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 3rd Semester Examination, 2019

CC6-MATHEMATICS

GROUP THEORY-I

Time Allotted: 2 Hours — Full Marks: 60

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GROUP-A

1. Answer any ***four*** of the following: $3\times4=12$

(a) Prove that a non-abelian group of order 10 must have a trivial centre.

(b) Give an example of an infinite abelian subgroup of a non-abelian group.

(c) Let G be a group in which $(ab)^3=a^3b^3$, $\forall a,b\in G$. Prove that $H=\{x^3:x\in G\}$ is a normal subgroup of G.

(d) Let $\beta=(1\ 2\ 3)(1\ 4\ 5)$, write β^{99} in cycle notation.

(e) Prove that there does not exist an onto homomorphism from the group $(\mathbb{Z}_6,+)$ to the group $(\mathbb{Z}_4,+)$.

(f) Find the number of elements of order 5 in $\mathbb{Z}_{15}\times\mathbb{Z}_5$.

GROUP-B

2. Answer any ***four*** of the following: $6\times4=24$

(a) (i) Prove that the set of all distinct left cosets of H in G and the set of all distinct right cosets of H in G have the same cardinality. 4

(ii) Give an example to show that a semigroup $(G,\circ)$ in which there is a left identity and right inverse, may not be a group. 2

(b) (i) If H be a subgroup of a cyclic group G, then the quotient group G/H is cyclic. 3

(ii) Let H be a normal subgroup of a group G, such that $O(H)=3$ and $[G:H]=10$. If $a\in G$ and $O(a)=3$, prove that $a\in H$. 3

(c) (i) Show that Klein's 4-group is isomorphic to $\mathbb{Z}_2\times\mathbb{Z}_2$. 4

(ii) Give an example of a group $(G,\circ)$ in which $O(a)$, $O(b)$ are infinite but $O(a\circ b)$ is finite, for $a,b\in G$. 2

(d) (i) Let G be a cyclic group of order n and is generated by a. Then prove that, for any positive integer r, a^r is also a generator iff $r < n$ and r is prime to n. 4

(ii) Show that the centre of a group $Z(G)$ is a subgroup of G. 2

(e) (i) Prove that the order of every subgroup of a finite group G is a divisor of the order of G. 4

(ii) Show that $\mathbb{Z}_6$ is not a homomorphic image of $\mathbb{Z}_{16}$. 2

(f) (i) Prove that upto isomorphism, there are only two groups of order 4. 4

(ii) Let $G \neq \{e\}$ be a group of order p^n, p is prime. Show that G contains an element of order p. 2

GROUP-C

3. Answer any *two* of the following: $12\times2 = 24$

(a) (i) Let G be a group in which $(ab)^3 = a^3b^3$, $\forall a, b \in G$. Show that 3+3

(A) $H = \{x^2 : x \in G\}$ is a subgroup of G

(B) $H = \{x^6 : x \in G\}$ is a subgroup of G.

(ii) Find all subgroups of S_3. Show that union of any two nontrivial distinct subgroups of S_3 is not a subgroup of S_3. 3+3

(b) (i) Let G and G' be two groups and $\phi : G \to G'$ be an onto homomorphism. Let $H = \ker\phi$. Then prove that the quotient group $G/H \cong G'$. 6

(ii) Prove that the set of matrices $S = \left\{\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z}\right\}$ forms a commutative subgroup of $GL(2, \mathbb{R})$. 6

(c) (i) Let H be a subgroup of a group G. Then prove that $K = \bigcap_{g \in G} gHg^{-1}$ is a normal subgroup of G. 4

(ii) Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 4 & 7 & 5 & 2 & 3 & 1 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 4 & 6 & 7 & 3 & 5 & 2 \end{pmatrix}$ be elements of S_7. 3+3+2

(A) Write α as a product of disjoint cycles

(B) Write β as a product of 2-cycles

(C) Is α^{-1} an even permutation?

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UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 3rd Semester Examination, 2019

CC7-MATHEMATICS

Time Allotted: 2 Hours | Full Marks: 60

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GROUP-A

Answer any *four* questions from the following $3\times 4 = 12$

1. If f is a non-negative continuous function on $[a, b]$ and $\int_a^b f(x)\,dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$. 3

2. Show that the integral $\int_0^\infty \frac{\cos x}{\sqrt{1+x^3}}\,dx$ converges absolutely by μ-test. 3

3. Show that the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$, where $f_n(x) = x^n$ is uniformly convergent on $[0, k]$, where $k < 1$. 3

4. Find the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ where $a_n = \frac{2^n}{n^2}$, $n = 1, 2, 3, \ldots\ldots$ and $a_0 = 0$. 3

5. Let $f(x, y)$ be defined over $S = [0 \le x \le 1,\ 0 \le y \le 1]$ by 3

$$f(x, y) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 3y^2 & \text{if } x \text{ is irrational} \end{cases}$$

Examine whether the iterated integrals $\int_0^1 dx \int_0^1 f(x, y)\,dy$ and $\int_0^1 dy \int_0^1 f(x, y)\,dx$ exist.

6. Prove that $\beta(m, n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\,dx$, $(m, n > 0)$. 3

GROUP-B

Answer any *four* questions from the following $6\times 4=24$

7. Let $[a, b]$ be a closed and bounded interval and $f:[a, b]\to \mathbb{R}$ be bounded on $[a, b]$. Then show that to each pre-assigned positive ε there corresponds a positive δ such that $U(P, f) < \int_a^{b} f + \varepsilon$ for all partitions P of $[a, b]$ satisfying $\| P \| < \delta$. 6

8. If $f(x)=\begin{cases} -x & \text{for } -\pi < x < 0 \\ 0 & \text{for } 0 < x < \pi \end{cases}$ 6

then show that Fourier series corresponding to $f(x)$ on $-\pi < x < \pi$ is $\frac{\pi}{4}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}+\sum_{n=1}^{\infty}\frac{(-1)^n \sin nx}{n}$.

9. Show that the Beta function $\beta(m, n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ exists if $0 < m,\ n < 1$. 6

10. Let $f:\mathbb{R}\to\mathbb{R}$ be uniformly continuous on $\mathbb{R}$. For each natural number n, let $f_n(x)=f(x+\frac{1}{n})$, $x\in\mathbb{R}$. Prove that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is uniformly convergent on $\mathbb{R}$. 6

11. Assuming the power series expansion for $\frac{1}{\sqrt{1-x^2}}$ as 6

$$\frac{1}{\sqrt{1-x^2}}=1+\frac{1}{2}x^2+\frac{1.3}{2.4}x^4+\frac{1.3.5}{2.4.6}x^6+\cdots\cdots$$

Obtain the power series expansion for $\sin^{-1}x$. Deduce that $1+\frac{1}{2.3}+\frac{1.3}{2.4.5}+\frac{1.3.5}{2.4.6.7}+\cdots=\frac{\pi}{2}$.

12. Use first Mean Value Theorem to prove that 6

$$\frac{\pi}{6}\le\int_0^{1/2}\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}\le\frac{\pi}{6}\frac{1}{\sqrt{1-k^2/4}} \text{ where } k^2<1.$$

GROUP-C

Answer any *two* questions from the following $12\times 2=24$

13 (a) State Riemann-Lebesgue lemma. 3

(b) Prove that $\int_0^{\infty}\sin(x^2)\,dx$ converges by Dirichlet's test. 3

(c) If a function f is bounded and integrable in $[0, a]$, $a>0$ and monotone in $(0, \delta)$, $0<\delta<a$ and $f(0+)=0$ then show that $\lim\limits_{n\to\infty}\int_0^a f(x)\left(\frac{\sin nx}{x}\right)dx=0$ (assume that $\int_0^\infty \frac{\sin x}{x}dx$ is convergent.) 6

14.(a) Let $f_n(x)=\log(n^2+x^2)$, $x\in\mathbb{R}$, show that 1+3+2

(i) each f_n is differentiable on $\mathbb{R}$

(ii) the sequence $\{f_n'\}_{n\in\mathbb{N}}$ is uniformly convergent on $\mathbb{R}$.

(iii) Is the sequence $\{f_n\}_{n\in\mathbb{N}}$ uniformly convergent on $\mathbb{R}$?

(b) State and prove Weierstrass' M-test for the convergence of a sequence of functions. 6

15.(a) If a power series $\sum\limits_{n=0}^{\infty}a_nx^n$ be neither nowhere divergent nor every where convergent, then show that there exists a positive real number R such that the series converges absolutely for all x satisfying $|x|<R$ and diverges for all x satisfying $|x|>R$. 6

(b) Give an example of a function f which is Riemann integrable without having a primitive. 2

(c) Find the sum of the series $\sum\limits_{n=0}^{\infty}(2^n+3^n)x^n$, indicating the range of validity. 4

16.(a) Find the Fourier Cosine series for the function f defined for $0\le x\le\pi$ as 3+3

$$f(x)=\begin{cases}\pi/3, & 0\le x<\pi/3\\ 0, & \pi/3<x<2\pi/3\\ -\pi/3, & 2\pi/3<x\le\pi\end{cases}$$

$f(\pi/3)=\pi/12$, $f(2\pi/3)=-\pi/12$

Find the sum of the series for $x=\pi/3$ and deduce that $1-\frac{1}{5}+\frac{1}{7}-\frac{1}{11}+\frac{1}{13}-\frac{1}{17}+\ldots\ldots=\frac{\pi}{2\sqrt{3}}$.

(b) Prove that the series $x^4+\frac{x^4}{1+x^4}+\frac{x^4}{(1+x^4)^2}+\cdots\cdots$ is not uniformly convergent on $[0,1]$. 6

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